FINANCIAL ENGINEERING:

Methods and cases

Paweł Kliber
Editor

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CONTENTS

PREFACE

This textbook contains materials for several courses that are taught in the Master Programme in Financial Engineering in the Poznań University of Economics and Business. The realm of financial engineering (or quantitative finance) is very broad and we have limited space, so we had to select some materials that we found the most important. As such topics we recognize investments, risk assessment and pricing. The chapters of this book are connected with these areas.

Chapter one covers methods used in the portfolio analysis. Starting from the classic Sharpe's Capital Asset Pricing Model, it moves to more refined and modern multifactor models of investment returns. The chapter contains real-life examples and cases from Polish and worldwide markets.

The second chapter describes basic methods of risk measurement in finance. It provides the definition of a risk and describes sources and types of risks one can encounter in financial institutions. It also provides the main measures of risk–both market risk as well as credit risk.

The third chapter provides an introduction to methods used in the derivative instruments pricing. The techniques of building formal models of financial markets are presented here. In the chapter basic notions connected with mathematical modelling in finance (such as arbitrage, risk-neutral measures and martingale pricing) are described. Due to lack of space only the discrete models (i.e. models with finite time horizon and sample space) are presented here.

Chapter four is devoted to corporate finance. It describes the main types of securities offered by companies to finance their economic activities. The main aims of issuing securities and methods of offering them are presented. The chapter contains also a description of practices from the Polish market and contains examples concerning this market.

The fifth chapter deals with modelling the term structure of interest rates. Starting from the basic concepts connected with time value of money, it introduces and describes various types of interest rates. The methods of estimating terms structure of interest rates from bonds' prices are presented here. The chapter ends with the description of the main models of term structure of interest rates that are used by central banks worldwide.

Chapter six provides broader view on the methods presented in the previous chapter and is related to the market practice. It contains information about the usage of term structure of interest rates in pricing swap instruments. The main swap instruments in the Polish financial markets are presented here. In the chapter it is shown that after the crisis of 2007-2009 more advanced methods, assuming the existence of many yield curves, are needed in practice.

The seventh chapter is devoted to hedge funds and their investment strategies. It presents an overview of the history of hedge funds and the reasons for their existence. Then it describes investment strategies used by such funds– in particular, strategies that make use of derivative instruments. The chapter ends with the examples of such strategies.

All the authors hope that this textbook will be helpful for the students of the financial engineering programme, but also for all who want to develop their knowledge in finance and, in particular, in quantitative methods used in finance.

CHAPTER 1
MULTIFACTOR MODELS: PORTFOLIO THEORY

The capital asset pricing model (CAPM), is one of the most elegant and appealing models in finance. This theory was independently developed by Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin (1966) on the fundamentals of Markowitz's efficient portfolio theory (Perold, 2004). Without exaggeration the CAPM has probably been one of the most useful and frequently used financial economic theories ever developed. It has also been widely discussed and questioned. In this chapter we will focus on the application of CAPM model to the Polish capital market, which according to Modern Index Strategy indexes delivered by Morgan Stanley Capital International (MSCI) classification, belongs to the European emerging markets.[1] First, we will start with single-factor models, CAPM, and then show how the most successful extensions used in the literature are employed for this particular market. We will also examine if multifactor models provide a better explanation for the behavior of stocks returns on the Warsaw Stock Exchange.

1.1. Capital asset pricing model

Let us start with the assumptions of the single and multifactor models. As the capital market theory is based on Markowitz's portfolio theory, the assumptions are nearly identical to those used in Markowitz's approach and may be summarized as follows (Reilly & Brown, 2002):

1. All investors are directed by the risk-return distribution. They want to obtain the market portfolio or any other portfolio that is on the efficient frontier. The choice of the exact portfolio depends on the personal utility function.
2. All investors have homogenous expectations.
3. Investors can borrow and lend money at the same risk-free rate.

[1] https://www.msci.com/market-classification.

4. All investors have the same investment horizon – the models are developed for a single hypothetical period.
5. There are no transaction costs, no taxes and no inflation. The investments are perfectly divisible. The capital markets are in equilibrium.

We start with the capital asset pricing model which represents the relationship between the risk and the expected rate of return. In the CAPM the expected excess return on any single risky asset, that is the difference between the expected return on the asset and the risk-free rate return, is proportional to the expected excess return on the market portfolio (Alexander, 2008). The excess return of a given asset depends on the excess market return with a special coefficient β:

$$E(R_i) - r_F = \beta_i (E(R_M) - r_F). \tag{1}$$

Without the expectations the formula is as follows:

$$R_i = r_F + \beta_i (R_M - r_F), \tag{2}$$

where $\beta_i = \dfrac{\text{cov}(R_i, R_M)}{\text{var}(R_M)} = \dfrac{\sigma_i \sigma_M \rho_{iM}}{\sigma_M^2}$,

r_F – is a risk-free rate,
R_i – is return from asset i,
R_M – is return from market portfolio,
M, σ_i, σ_M – stand for standard deviations of returns of asset i and the market portfolio,
ρ_{iM} – describes the correlation coefficient between returns.

On the basis of this return-generating model one easily obtain the required (proper) return from the investment in asset i given the actual market conditions. One can compare the required rate of return with the estimated rate of return to asses if the asset is overvalued, undervalued or properly valued. The difference between expected returns (1) and realized returns (2) is due to errors that appear when the expectations are related to the realized values:

$$e_i = R_i - r_F - \beta_i (R_M - r_F). \tag{3}$$

The beta coefficient in this approach represents assets' sensitivity to the market portfolio changes and as such it is perceived as a measure of a systematic risk. As it relates the covariance to the variance of market portfolio, it is also a standardized measure and thus can be compared across the stocks listed on a given market. If beta is higher than 1, then the asset is said to be aggressive, it means it both grows and decreases faster than the market portfolio. We may also say that the asset has a higher standardized systematic risk than the market and thus it is more volatile than the overall market portfolio. If the beta is equal to one, then the portfolio behaves as a market portfolio. Beta in the interval (0,1) means that the asset grows more slowly than the market and decreases also slowly; such an asset has lower volatility than the market. The most interesting is a negative beta case: it happens when the asset returns are changing in different direction than the market portfolio returns. There are many approaches to calculate beta in the real world and we discuss them in subchapter 1.2.2.

The Security Market Line (SML) is derived from the CAPM (1) model, where the expected return $E(R_i)$ depends on the beta coefficient, β. The SML is often used for valuation and allows to examine if a given asset is undervalued, overvalued or properly valued. Based on the actual characteristics of the market, the return of the market portfolio, beta of an instrument and the risk-free rate, one is able to examine if the asset returns fit the SML. In case the asset return is above the SML, this instrument is undervalued, whereas if the return is below the SML, it is overvalued. However, in the equilibrium, all single assets should "lie" on the SML.

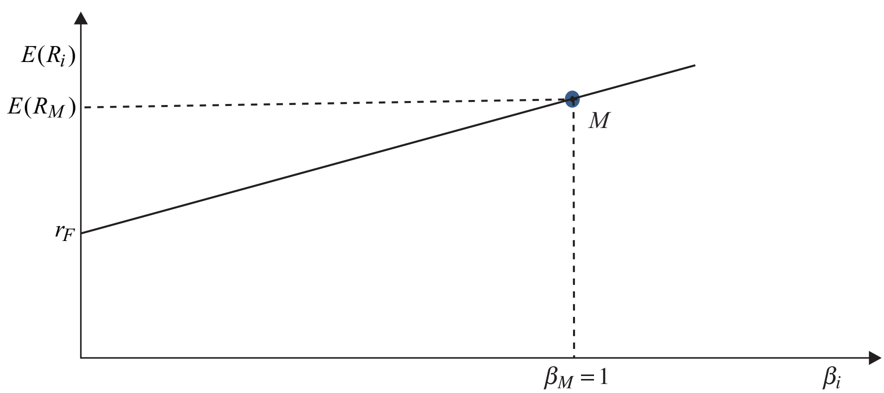


Figure 1. The Securities Market Line

An investor considers not only the return, but also risk of the stock. In the CAPM approach risk is measured as the variance of the returns and based on Eq. (2) is expressed in the following way:

$$\operatorname{var}(R_i) = \operatorname{var}\left(r_F + \beta_i\left(R_M - r_F\right) + e_i\right). \tag{4}$$

Taking into account that r_F and β are fixed:

$$\operatorname{var}(R_i) = \beta_i^2 \operatorname{var}\left(R_M - r_F\right) + \operatorname{var}(e_i) + 2\operatorname{cov}\left(\beta_i\left(R_M - r_F\right)(e_i)\right). \tag{5}$$

As we assume also that market portfolio return should be not correlated with the error term, we obtain:

$$\operatorname{var}(R_i) = \beta_i^2 \operatorname{var}(R_M) + \operatorname{var}(e_i). \tag{6}$$

Thus the total variance is decomposed into a systematic variance (measured by the beta coefficient) and a non-systematic (idiosyncratic) variance. The systematic risk is due to the market as a whole and is non-diversifiable, while the non-systematic risk can be diversified by increasing the number of the assets in the portfolio.

Since Ang, Hodrick, Xing and Zhang (2006), there is an ongoing discussion about the idiosyncratic volatility *IVOL* puzzle. This phenomena appears when the performance of stocks with a low idiosyncratic risk outperforms that of stocks with a high idiosyncratic risk. Stambaugh, Yu and Yuan (2015) find that the relationship between idiosyncratic volatility and return is negative only among overpriced stocks, while among underpriced stocks this relationship is positive. However, Zaremba (2018) shows that this feature comes out of the mathematical properties of return distributions.

Problems and solutions

Problem I

Consider the following characteristics of market and stock X: the risk-free rate is 2%, the expected rate (based on fundamental value) of return from market portfolio is 6% and the risk measured by its standard deviation is 4%, while for X asset it is 5%. The correlation coefficient of asset X' returns with the market portfolio returns is +0.2. The stock is priced at 100 EUR and is supposed to be worth 107 within a year. Calculate beta for asset X and find out if stock X is undervalued or overvalued.

Solution

The expected return for stock X is: $E(R_i) = (107 - 100)/100 = 7\%$
Beta for stock X: As $\rho_{MX} = 0.2$,

$$\beta = \frac{\operatorname{cov}(R_X, R_M)}{\operatorname{var}(R_M)} = \frac{\sigma_M \sigma_X \cdot \rho_{MX}}{\sigma_M^2} = \frac{\sigma_X \cdot \rho_{MX}}{\sigma_M} = \frac{5 \cdot 0.2}{4} = 0.25$$

(and $\operatorname{cov}(R_X, R_M) = \sigma_M \sigma_X \cdot \rho_{MX} = (4 \cdot 5)/0.2 = 100$).

The return based on the SML is:

$$E(R_X) = r_F + \beta\left(E(R_M) - r_F\right) = 2 + 0.25 \cdot (6-2) = 3\%.$$

As our expected rate of return is 7%, and this is higher than 2.5% from SML, asset X is undervalued.

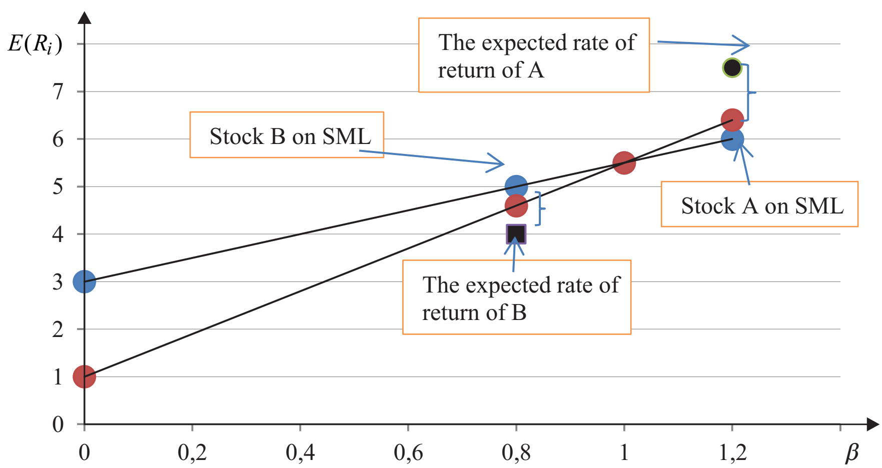
Problem II

An analyst expects a risk-free rate of 3%, a market return of 5.5% with a risk measured with standard deviation of 3%. The characteristics for stocks A and B are shown below:

Stock	beta	Standard deviation (%)	Expected rate of return (%)
A	1.2	5	7.5
B	0.8	2.5	4

1. Draw SML and find out if the stocks are fairly valued by the market (under/overvalued)?
2. Will your conclusion change if the risk-free rate decreases to 1%?

Solution



Blue bullets are for the SML line with $r_F = 3\%$, while red bullets are for the SML line with $r_F = 1\%$.

The black bullet depicts the expected rate of return of A–the difference between the black and blue (red) bullet for a given value of beta shows that asset A is undervalued. On the contrary, the black square depicts the expected rate of return of B that irrespectively of the value of risk-free rate is overvalued.

The results of calculations are presented below:

Stock	beta	Standard deviation (%)	Expected rate of return	SML valuation under $r_F = 3\%$ (%)	SML valuation under $r_F = 1\%$ (%)
A	1.2	5	7.5	**6**	**6.4**
B	0.8	2.5	4	**5**	**4.6**
M	1	3	5.5		

With respect to the second question, the conclusion is not changing as the risk-free rate moves from 3% to 1%–both the undervalued and overvalued asset are still under- or overvalued.

The SML requires that the risk-free rate is known. As it is often controversial which rate would be the best proxy for the risk-free rate and should be taken into account, a solution to this problem is to use the market model which does not contain a risk-free rate.

1.2. The characteristic line–market model

1.2.1. Model specification

The market model, also called the characteristic line of the security, for an asset *i* has the following specification:

$$E(R_i) = \alpha_i + \beta_i E(R_M) + e_i, \tag{7}$$

where $\alpha_i = (1 - \beta_i) r_F$, α and β are parameters, and *e* is an error term.

The assumptions for the characteristic line usually are the following:

1. $E(e_i) = 0$
2. Variance of the error term is homoscedastic: $\operatorname{var}(e_i) = \sigma_{e_i}^2$.
3. The error terms are not correlated: $\operatorname{cov}(e_i, e_j) = 0$ for each $i \neq j$.
4. The covariance of error term with market portfolio return is zero: $\operatorname{cov}(e_i, R_M) = 0$ for each *i*.

Without the expectations the market model is the following:

$$R_i = \alpha_i + \beta_i \left(R_M\right) + e_i, \tag{8}$$

and the variance of the return is given by

$$\operatorname{var}\left(R_i\right) = \operatorname{var}\left(\alpha_i + \beta_i \left(R_M\right)\right) + \operatorname{var}\left(e_i\right), \tag{9}$$

or after transformation and in different notation:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2. \tag{10}$$

The last two equations show that risk is decomposed into two parts: a systematic risk and a non-systematic one. The former is dependent on the market where the asset is listed, and the latter represents the risk of the asset itself.

One can easily present the characteristic line for the portfolio as follows:

$$R_P = \alpha_P + \beta_P \left(R_M\right) + e_P, \tag{11}$$

where

$$\alpha_P = \sum_{i=1}^{n} x_i \alpha_i, \ \ \beta_P = \sum_{i=1}^{n} x_i \beta_i, \ \ R_P = \sum_{i=1}^{n} x_i R_i, \ \ e_P = \sum_{i=1}^{n} x_i e_i.$$

The variance of the portfolio return is given by:

$$\operatorname{var}\left(R_P\right) = \operatorname{var}\left(\alpha_P + \beta_P R_M\right) + \operatorname{var}\left(e_P\right), \tag{12}$$

$$\operatorname{var}\left(R_P\right) = \beta_P^2 \operatorname{var}\left(R_M\right) + \operatorname{var}\left(e_P\right), \tag{13}$$

The last component in the previous formula is calculated as follows:

$$\operatorname{var}\left(e_P\right) = \mathbf{x}'\mathbf{V_e}\mathbf{x}, \tag{14}$$

where $\mathbf{x}$ is a vector of weights of assets and $\mathbf{V_e}$ is the variance-covariance matrix for error terms. In case the errors from the characteristic lines of different stocks in the portfolio are not correlated, the formula for non-systematic variance can be reduced to:

$$\operatorname{var}\left(e_P\right) = \sum_{i=1}^{n} x_i^2 \operatorname{var}\left(e_i\right).$$

The parameters of the market model (characteristic line) are estimated usually with ordinary least squares. This is achieved by obtaining the best fit to a regression line through scatter plot returns for the individual risky asset and returns for the market portfolio over some declared period. The number of observations and their frequency (monthly, weekly or daily) may vary. As the length of the sample period is concerned, a five-year period is a good and common choice. Reilly and Brown (2002) state that Value Line Investment Services obtains market models using weekly returns for five years (260 weekly observations), while Merrill Lynch, Pierce, Fenner & Smith use monthly returns for the same period (60 monthly observations). Also Thomson Reuters shows the beta coefficient based on 5-year monthly observations.

The second issue is also undefined: there is no unified correct time interval for analysis. This is always a trade-off between the number of observations and the empirical distribution of returns in the sample: it is a stylized fact in financial econometrics that due to the aggregation, monthly data are more often normally distributed than weekly data.

1.2.2. Examples of the market model estimation

Let us consider the characteristic lines for returns of 12 stocks listed on the Warsaw Stock Exchange (WSE). These stocks are (in an alphabetical order, with tickers in the brackets): Agora (AGO), Assecopol (ACP), Boryszew (BRS), BZ WBK (BZW), Bank Handlowy (BHW), KGHM (KGH), Kęty (KTY), mBank (MBK), Netia (NET), Orange (OPL), PEKAO S.A. (PEO) and PKN Orlen (PKN). The estimates of the parameters are obtained within the market models for monthly data for each stock separately. The sample period consists of 60 monthly observations, from 2012.02 to 2017.01 (data are attached in Data_book_monthly.xlsx).

Table 1 presents the estimates of alfa and beta parameters as well as standard deviations of error terms (s) and determination coefficient, R^2, for 12 regressions. The beta coefficient varies from 0.19 (NET) to 1.32 (KGH)–our sample includes stocks that react to the market changes with different sensitivity. All but one beta parameters are statistically significant at $\alpha = 0.05$. This exception is the estimate for NET returns, that is not different from 0. The α parameter is statistically significant only in one instance, for KTY–its positive value signifies that the stock might be undervalued. The determination coefficient R^2 ranges from 0.02 (NET) to 0.5 (BHW) showing that in most cases the variability of stock returns is rather inadequately explained by the single-factor model.

Next we calculate the systematic and non-systematic risk for the individual stocks and portfolio consisting of these assets in three scenarios: equally-weighted portfolio (scenario 1), portfolio minimizing the non-systematic risk (scenario 2) and minimizing the total risk (scenario 3). The results are presented in Table 2.

Table 1. Estimates from the market models for 12 stocks listed on the WSE

	β	α	R^2	σ (%)
AGO	**1.17**	0.30	0.19	9.29
ACP	**0.60**	0.47	0.18	5.04
BRS	**1.03**	0.05	0.20	7.97
BZW	**0.99**	0.54	0.34	5.37
BHW	**1.29**	0.11	0.50	4.98
KGH	**1.32**	–0.19	0.27	8.33
KTY	**0.79**	**2.42**	0.18	6.47
MBK	**1.26**	–0.01	0.45	5.41
NET	0.19	0.15	0.02	5.60
OPL	**1.03**	–1.36	0.18	8.61
PEO	**1.07**	–0.35	0.49	4.26
PKN	**1.22**	1.34	0.37	6.15

Note: The bolded values of parameters depict the statistical significance at significance level $\alpha=0.05$.

Table 2. Market model: The risk in equally-weighted portfolio (scenario 1)

	β	α	R^2	σ (%)	Systematic risk	Non-systematic risk	Total risk	x_i	$(x_i)^2\sigma^2$
AGO	1.17	0.30	0.19	9.29	20.13	86.24	106.37	0.08	0.60
ACP	0.60	0.47	0.18	5.04	5.39	25.42	30.81	0.08	0.18
BRS	1.03	0.05	0.20	7.97	15.55	63.47	79.02	0.08	0.44
BZW	0.99	0.54	0.34	5.37	14.40	28.89	43.28	0.08	0.20
BHW	1.29	0.11	0.50	4.98	24.56	24.79	49.36	0.08	0.17
KGH	1.32	–0.19	0.27	8.33	25.63	69.41	95.04	0.08	0.48
KTY	0.79	2.42	0.18	6.47	9.29	41.89	51.18	0.08	0.29
MBK	1.26	–0.01	0.45	5.41	23.55	29.32	52.87	0.08	0.20
NET	0.19	0.15	0.02	5.60	0.51	31.33	31.83	0.08	0.22
OPL	1.03	–1.36	0.18	8.61	15.64	74.08	89.72	0.08	0.51
PEO	1.07	–0.35	0.49	4.26	16.87	18.15	35.02	0.08	0.13
PKN	1.22	1.34	0.37	6.15	21.82	37.77	59.58	0.08	0.26
Portfolio	**1.00**				**14.70**	**3.69**	**18.39**	**1.00**	

If we assume the equal weights for the assets in the portfolio, then for our sample the estimate of beta portfolio is equal to 1 (which is rather accidental) and the total risk accounts for 18.39. The estimated variance of the market portfolio for the given period is 14.76. The systematic risk accounts for 80% of the total risk. The total risk of the portfolio is much lower than the total risk of an individual asset in the portfolio–this is due to the diversification effect. It is clearly seen when one looks at the non-systematic risk: for the portfolio it equals 3.69, while for the single assets it ranges from 18.15 to 86.24.

In the next two scenarios we use Solver to find the minimum variance portfolio. Table 3 presents the risk estimates in scenario 2 and 3. In the former the non-systematic risk is minimized. Note that this portfolio is nicely balanced with the total risk of 17.29. The overall beta in this case is 0.7 and the systematic risk accounts for 83% of the total risk. In the latter scenario 3 we minimize the risk focusing on the total risk. As a result we obtain the lowest value of total risk out of our three scenarios equal to 15.26, but the systematic risk accounts for 68% of the total risk. Note that in the portfolio the assets that dominate have beta coefficients lower than 1 (e.g. ACP or NET). Again our portfolios are well-diversified as the non-systematic risk is much lower for the portfolio than for the single stocks, and, as a result, the total risk of the portfolio is much lower than in Scenario 1 and 2.

Table 3. Market model: The risk estimates (scenario 2 and 3)

	Scenario 2				**Scenario 3**			
	Systematic risk	**Non-systematic**	**Total risk**	x_i	**Systematic risk**	**Non-systematic**	**Total risk**	x_i
AGO	20.13	86.24	106.37	0.03	20.13	86.24	106.37	0.02
ACP	5.39	25.42	30.81	0.12	5.39	25.42	30.81	0.22
BRS	15.55	63.47	79.02	0.05	15.55	63.47	79.02	0.04
BZW	14.40	28.89	43.28	0.10	14.40	28.89	43.28	0.10
BHW	24.56	24.79	49.36	0.12	24.56	24.79	49.36	0.02
KGH	25.63	69.41	95.04	0.04	25.63	69.41	95.04	0.01
KTY	9.29	41.89	51.18	0.07	9.29	41.89	51.18	0.10
MBK	23.55	29.32	52.87	0.10	23.55	29.32	52.87	0.03
NET	0.51	31.33	31.83	0.09	0.51	31.33	31.83	0.28
OPL	15.64	74.08	89.72	0.04	15.64	74.08	89.72	0.03
PEO	16.87	18.15	35.02	0.16	16.87	18.15	35.02	0.12
PKN	21.82	37.77	59.58	0.08	21.82	37.77	59.58	0.03
Portfolio	**14.35**	**2.93**	**17.29**	**1.00**	**10.31**	**4.96**	**15.26**	**1.00**

The characteristic lines for two assets, KGH and ACP, on the basis of monthly returns in the period 2012.02-2017.01 are depicted on Figure 2. KGH is more aggressive as its beta is equal to 1.32 for the given period, while ACP is a defensive stock with beta of 0.6. Each square (KGH) and diamond (ACP) represents the combinations of market returns (*X* axis) and individual stock returns (*Y* axis). The variability of the returns is quite remarkable–they only rarely lie

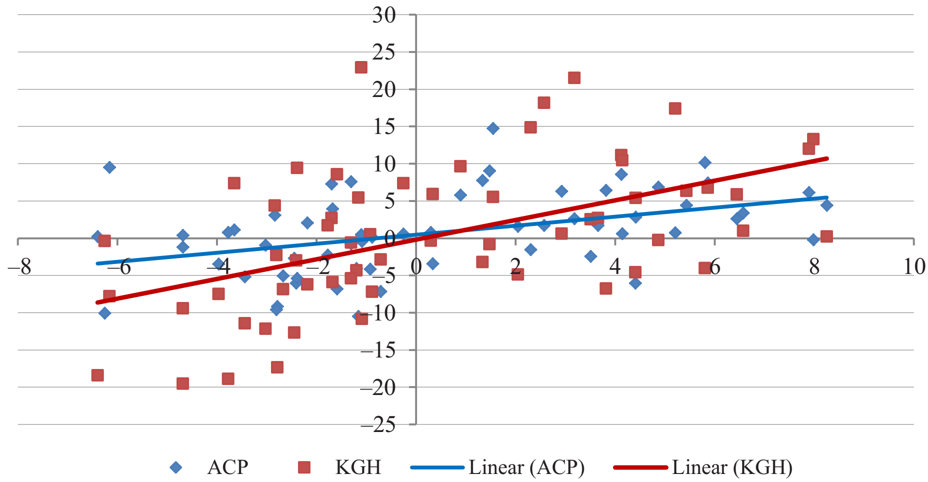


Figure 2. The estimates of characteristic lines for ACP and KGH stocks

on the characteristic lines, thus causing low values of determination coefficients, 27% for KGH versus 18% for ACP.

1.2.3. Stability of beta

Tests of the CAPM indicate that the beta coefficients for individual securities are often not stable within longer time periods, although the portfolio beta are generally stable, assuming that sample periods are long enough and trading volume is adequate. We will provide some empirical evidence for the hypothesis of beta stability for individual stocks. This issue is already discussed in the literature (Brooks, Faff, & McKenzie, 1998; Faff, Millier, & Millier, 2000; Dębski, Feder-Sempach, & Świderski, 2016; Będowska-Sójka, 2017).

We estimate the beta coefficient in the moving window of 60 monthly data starting from 2000.01 to 2017.01. As the moving window is used, the estimates of beta coefficients are obtained from 2004.12. We get altogether 145 betas for each return series. The descriptive statistics for the beta estimates presented in Table 4 show the mean, standard deviations, minimum and maximum values.

Table 4. The descriptive statistics of the estimates of beta parameters for 12 stocks

Variable	Min	Mean	Max	Standard deviation
AGO	0.57	1.09	1.53	0.23
ACP	0.52	0.85	1.71	0.26
BRS	1.11	1.66	2.42	0.33

Table 4 – cont.

Variable	Min	Mean	Max	Standard deviation
BZW	0.52	1.10	1.46	0.27
BHW	0.41	0.94	1.28	0.26
KGH	1.23	1.47	1.78	0.11
KTY	0.54	0.79	1.01	0.07
MBK	1.08	1.43	1.67	0.17
NET	0.23	0.80	2.40	0.61
OPL	0.19	0.64	1.27	0.34
PEO	0.95	1.16	1.42	0.13
PKN	0.86	1.03	1.30	0.12

Figure 3 presents the beta coefficients within the sample period for 12 stocks. The visual inspection indicates that beta estimates are time-varying, although the differences between beta coefficients differ from one stock to another. There is no common tendency in the beta' dynamics for the considered stocks.

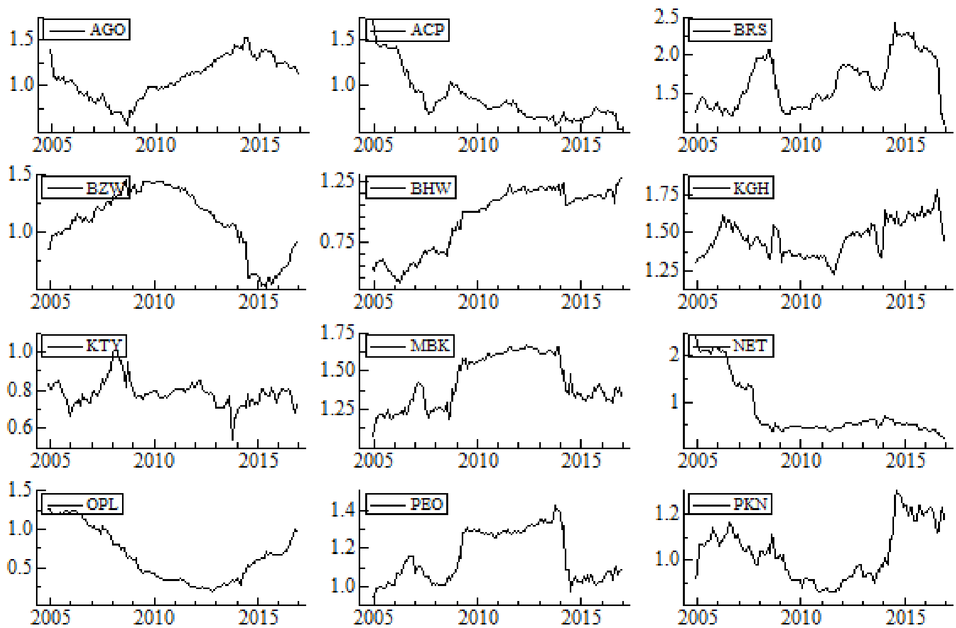


Figure 3. The monthly beta coefficients for 12 stocks from 2004.12 till 2017.01

1.2.4. Market portfolio: The most important indices

The market portfolio is an efficient portfolio consisting of all risky assets available on the market. The weights of these assets depend on the market values of the constituents of the portfolio. The concept of market portfolio is purely theoretical, thus its empirical representation is not obvious. Generally, for a given stock market the broadest stock index is used as a proxy for unknown market portfolio. Usually it is assumed that the proxy is highly correlated with the market portfolio. If the market portfolio is specified mistakenly, then the beta coefficients computed for individual assets and managed portfolios are wrong, and, as a result, the inference on under- or overvaluation is wrong, too. There are many different proxies that might be used for the market model and the choice of the proxy will definitely impact the results.

Although this problem is recognized widely in the literature, it is usually assumed that the national stock market indexes provide exposure to individual countries. Each stock exchange has at least one, the most important index that measures the weighted value of a country's stock market. Sometimes these indices consist entirely of the large-cap stocks (e.g. the Dow Jones Industrial Average or, in a broader sense, Standard and Poor 500 in the United States, DAX in Germany, FTSE100 in Great Britain or CAC40 in France). Other indices will consist of all the equities that are traded on the market (e.g. FTSE All-Share Index in Great Britain, S&P TSX Composite Index in Canada, WIG Index in Poland) if only the stocks satisfy the relevant base criteria.

As the market became a global scene, in the comparative studies across different countries the global stock market indices are often used. The dominance of the American and European stock markets has passed and the stocks markets in Asian and Latin America countries have become more important in recent decades. The markets form Japan, South Korea, China, India or Brazil are among the top exchanges in the world. The global indices track stocks from all around the world. The most well-known ones that cover developed markets include MSCI World Index, FTSE-All World Index, S&P Global 100, STOXX Global Indices, Dow Jones Global Titans 50 and Russell Global Index (www.thebalance.com). There are also the regional indices that cover Asian, European and Latin America regions as e.g. S&P Asia 50 Index, S&P Europe 350 Index and S&P Latin America 40 Index. MSCI divide regions into 3 broad categories: Americas, Europe & Middle East and Asia. Different indices are used in the emerging and the frontier markets–here the examples are MSCI Emerging Markets Index or MSCI Frontier Markets Index (www.msci.com). STOXX regional indices are devoted to Americas (STOXX Americas), Asia/Pacific (STOXX Asia), Europe (Euro STOXX or STOXX Europe) and Africa (STOXX Africa 90). Additionally they offer about 70 indices for individual countries (www.stoxx.com). The broad

STOXX indices include STOXX Developed Indices and STOXX Emerging Indices. Some indices are based on the size, industry, dividend, etc.

With such a diversity of the stock indices available as the potential proxy for an unknown market portfolio, the decision which index to choose and which would be the best one, is difficult.

Problems and solutions

Problem

Calculate the systematic risk of the portfolio consisting of three assets. The variance of market portfolio is 2.09. The characteristics of these assets are the following:

Asset	Correlation with R_M	Variance of returns	Weights
A	0.6	3.22	0.2
B	0.2	7.15	0.5
C	–0.1	4.23	0.3
Market	1	2.09	

Solution

As we want to calculate the systematic risk of the portfolio, we will need beta coefficient (see eq. 13). Thus first we calculate the beta for each asset separately.

$$\beta = \frac{\operatorname{cov}(R_i, R_M)}{\operatorname{var}(R_M)} = \frac{\sigma_M \sigma_i \rho_{Mi}}{\sigma_M^2} = \frac{\sigma_i \rho_{Mi}}{\sigma_M},$$

$$\beta_A = \frac{\sqrt{3.22} \cdot 0.6}{\sqrt{2.09}} = 0.74,$$

$$\beta_B = \frac{\sqrt{7.15} \cdot 0.2}{\sqrt{2.09}} = 0.37,$$

$$\beta_C = \frac{\sqrt{4.23} \cdot (-0.1)}{\sqrt{2.09}} = -0.14,$$

$$\beta_P = \sum_{i=1}^{n} x_i \beta_i = 0.2 \cdot 0.74 + 0.5 \cdot 0.37 + 0.3 \cdot (-0.14) = 0.29,$$

$$\beta_P^2 \sigma_M^2 = 0.29^2 \cdot 2.09 \approx 0.18.$$

The systematic risk of the portfolio is approximately 0.18.

1.3. Multifactor models

The market model (as well as SML model) is an example of a single-factor model that allows to decompose the risk into a market (systematic) and a firm-specific (non-systematic) component. However, this decomposition might ignore some factors, as it assumes that all stock return series respond to the same market factor. Thus multifactor models that allow for different securities to be sensitive to different factors can provide a better description of the returns.

The general specification for the multifactor model may be expressed in the following way:

$$R_i = \alpha_i + \beta_{i1} f_1 + \ldots + \beta_{ik} f_k + e_i, \tag{16}$$

where the assumptions are similar to these given in the market model (eq. 7). The beta coefficients are called factor sensitivities (or factor loadings/betas). In the literature the factors are related to the macroeconomic variables (e.g. GDP or interest rates) or market premiums based on the various characteristics (such as size, value/income stocks, liquidity, etc.). Below a few widely-used models are presented: Fama and French model (1992), Carhart model (1997) and 5-factors Fama and French model (2015).

1.3.1. Fama and French (1992)

Fama and French (1992) proposed three factor models in which they added to the market factor in the plain CAPM two additional risk factors:

$$R_{it} - r_{Ft} = b_{i0} + b_{i1}\left(R_{Mt} - r_{Ft}\right) + b_{i2} SMB_t + b_{i3} HML_t + e_{it}, \tag{17}$$

where $R_{Mt} - r_{Ft}$ accounts for the excess returns to a broad market index (market portfolio), SMB_t (*small minus big*) is the excess return obtained from portfolios of small-cap versus portfolios of large-cap stocks and thus reflects size premium, whereas the value effect, HML_t (*high minus low*) is the difference of returns between portfolios of value-oriented stocks (with high book-to-market ratio) versus growth-oriented portfolios (with low book-to-market ratio). Shortly, the value effect is the superior performance of stocks with a low price to book compared with stocks with a high price to book.

In the further presentation we use the data available on the web-site of Adam Zaremba, PhD.[2] (http://adamzaremba.pl/downloadable-data/) with monthly data series on Fama–French (1992) three factor model of and Carhart's (1997) four factor model with momentum for the Polish stock market.

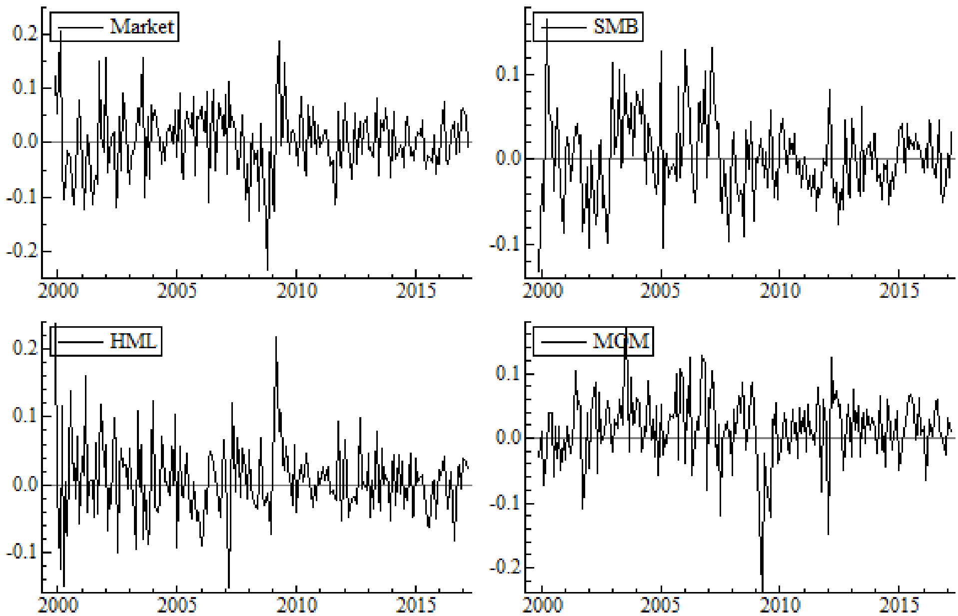


Figure 4. The factors estimated for the Polish stock market

Note: There are 4 premiums presented: Market stand for market premium, SMB for small minus big, HML for high minus low and MOM for momentum from Carhart model (1997).

Figure 4 presents the factors estimated for the Polish stock market. The most striking result is that the premiums are very volatile within the time. There are periods of the existence of the premium for the small stocks (e.g. in the period of 2003-2004) and periods for which the premium for big stocks appears (e.g. 2011).

Table 5 presents the estimates from the Fama and French three factor model for 12 stocks listed on the Warsaw Stock Exchange in the period 2000.01-2017.01. The market beta is statistically significant for all the stocks and in most cases different from 1 (the exceptions are: BZW and PEO). As the SMB risk premium is considered, it is statistically significantly different from zero only in five out

[2] The author is grateful to prof. Adam Zaremba for the permission to use this dataset.

of twelve stocks. If the parameter b_{i2} is positive (and this is the case for BRS and KTY) the stocks are "recognized" or valued by the market as the small stocks. In fact, if we look at the capitalization of these stocks presented in Table 5, they may be considered as smaller companies (but not the smallest ones). If the parameter b_{i2} is negative (as it is for NET, OPL and PEO), it shows that stocks are considered by the market as big ones. It is obviously true for PEO and OPL, which are large-cap companies, but not that much for NET, as its capitalization is smaller than KTY. One should note that the estimation is done on the 17-year sample, while the capitalization comes from the end of this period. That might explain the lack of coherence in this aspect.

When the parameter b_{i3} is considered, the positive values suggest that a stock is valued as an income stock, as HML is the differential between high book-to--market and low book-to-market stocks. Out of our 12 stocks only three have statistically significant parameter, BHW, KGH and PKN–all of them are thus regarded as the income stocks. The negative values are not statistically significant.

The Fama and French three factor models do describe the returns better than the single factor market model. The comparisons of the determination coefficients for the individual regressions presented in Table 2 and Table 5 show it definitely: the R^2 increases in the latter approach for all stocks, even if in the best case only two out of three parameters for the given factors are statistically significant.

Table 5. The estimates of the parameters in Fama and French three factor model

	AGO	**ACP**	**BRS**	**BZW**	**BHW**	**KGH**
b_{i0}	–0.0035	0.0042	**0.0237****	**0.0108****	0.0024	**0.0117***
b_{i1} (Market)	**1.2082*****	**1.1514*****	**1.4682*****	**1.0599*****	**0.8346*****	**1.2839*****
b_{i2} (SMB)	–0.1263	–0.0590	**1.2048**	–0.0942	0.0254	0.1884
b_{i3} (HML)	–0.0338	–0.1809	0.2365	0.0125	**0.2031****	**0.4958*****
R^2	0.41	0.34	0.30	0.49	0.40	0.47
capitalization	0.691	0.815	2.25	36.93	10.56	20.6
P/E	*na*	8.15	11.58	16.71	21.01	*na*
P/BV	0.69	0.7	1.96	1.69	1.57	1.16
	KTY	**MBK**	**NET**	**OPL**	**PEO**	**PKN**
b_{i0}	**0.0104***	0.0064	–0.0009	–0.0032	0.0059	0.0052
b_{i1} (Market)	**0.7970*****	**1.3118*****	**1.1758*****	**0.8114*****	**1.0569*****	**0.8963*****
b_{i2} (SMB)	**0.2248***	0.1557	**–0.5687****	**–0.333*****	**–0.1978****	–0.1005
b_{i3} (HML)	–0.0256	0.1594	–0.2799	–0.0560	0.0314	**0.2179*****

Table 5 – cont.

	AGO	ACP	BRS	BZW	BHW	KGH
R^2	0.29	0.53	0.25	0.38	0.62	0.50
capitalization	3.4	19.18	1.88	7.82	34.09	40.91
P/E	13.89	17.57	52.8	*na*	13.74	6.11
P/BV	2.57	1.34	1.05	0.79	1.47	1.27

Note: The parameters presented are from the three-factor Fama and French model. The stars *, **, *** denote statistical significance at α 0.1, 0.05 and 0.01 respectively. The date for capitalization (in billions of Polish zloty), price-to-earnings (P/E) and price-to-book value (P/BV) are from the end of 2017.

Source: www.stooq.pl.

1.3.2. Momentum and further extensions

Two of the most popular strategies for investments are momentum and contrarian. The first one refers to the tendency of assets with good past performance to outperform in the future, and those with bad performance to underperform in the future. The second one aims to form a portfolio based on the assets that are not yet recognized by the market as promising ones: the best time to buy a stock is when its prices are in a downturn, and to sell – when the prices are at the top. The extension of the Fama and French model proposed by Carhart (1997) includes momentum as is a fourth common risk factor that accounts for the tendency for firms with positive past returns to produce positive returns in future, while for firms with negative returns one should expect negative returns in the future. Momentum factor, henceforth *MOM*, is estimated by taking the average return to a set of stocks with the best performance over the prior year minus the average return to stocks with the worst returns. Thus the model is following:

$$R_{it} - r_{Ft} = b_{i0} + b_{i1}\left(R_{Mt} - r_{Ft}\right) + b_{i2}SMB_t + b_{i3}HML_t + b_{i4}MOM_t + e_{it}. \quad (18)$$

Typically, momentum factor sensitivity for the momentum variable is positive, if momentum holds. Carhart (1997) shows that the inclusion of the momentum factor increases the determination coefficient in the regression by 15%. The extensive international studies on momentum anomaly are provided e.g in Asness, Moskowitz and Pedersen (2013) and Zaremba (2018).

In Table 6 we present the estimates from four factor models for these stocks in our sample for which the momentum appeared to be significant.

The momentum factor is statistically significant only for AGO, ACP, BZW and PKN. Surprisingly, in two of them, AGO and ACP, the parameter has a negative value contradicting the momentum effect. The remaining two, BZW and

PKN, display positive values and thus the momentum effect is observed. Last but not least, the inclusion of momentum does not improve determination coefficient in the regressions significantly.

Table 6. The estimates of four factor Carhart models for stocks listed on the Warsaw Stock Exchange

	AGO	ACP	BZW	PKN
b_{i0}	0.00	0.01	0.01	0.00
b_{i1} (Market)	**1.17*** **	**1.11*** **	**1.08*** **	**0.93*** **
b_{i2} (SMB)	–0.10	–0.04	–0.09	–0.10
b_{i3} (HML)	–0.18	**–0.30***	0.11	**0.34*** **
b_{i4} (MOM)	**–0.32** **	**–0.26***	**0.18***	**0.25*** **
R^2	0.42	0.36	0.50	0.52

Note: The parameters presented in the Table are from 4 factor Carhart models. The stars *, **, *** denote statistical significance at α level of 0.1, 0.05 and 0.01 respectively.

Another extension of the Fama and French (1993) model is added by its authors. In Fama and French (2015) they include two new factors already known as the quality factors and thus created a five factor model. Two new factors are: investments and profitability. The former implies that stocks of companies with the high total asset growth should have lower average returns, while the latter implies that stocks with a higher operating profitability would perform better. This approach ignores well-established momentum factor and, as a new one, is still under examination (Fama & French, 2015).

1.3.3. Liquidity premium

The above mentioned models are widely used in the literature. However, they do not include one factor that seems to provide useful information for the investors trading in emerging markets: liquidity premium (Lischewski & Voronkowa, 2012). The most common measure for it is Amihud's (2002) illiquidity that is related to Kyle's (1985) price impact measure. The illiquidity $ILLIQ$ is calculated as follows:

$$ILLIQ_{it} = |R_{it}| / (P_{it} \cdot VOL_{it}), \quad (19)$$

where R_{it} and P_{ti} are the return and the price of asset i on day t, while VOL_{tt} is the number of shares of asset i traded during the day t. Thus in the denominator we obtain the trading volume of a given asset. $ILLIQ$ measures the relative price change that is caused by a given trading volume. Usually the ratio is obtained for the daily data and then aggregated to the monthly (or yearly) measures. Amihud (2002) examines one-period lagged $ILLIQ$ as a factor in the cross-section study for NYSE stocks where other variables such as beta, size, volatility, dividend yield and past returns (for momentum effect) are included. The results show that illiquidity has a positive effect on stock returns.

Other measures of liquidity that are based on daily data and might be easily calculated include: volume over volatility (Fong, Holden, & Tobek, 2017) and adjusted quoted close spread (Chung & Zhang, 2014; Będowska-Sójka, 2018). Volume over volatility is based on the idea that for liquid stocks a given level of volume will cause lower distortions in price than for illiquid stocks. The distortion in prices are proxied with the range between the high and the low prices observed within a given day, whereas volume is square rooted in order to minimize the impact of extreme values of volume. The volume over volatility VoV_t is calculated as follows:

$$VoV_t = \frac{\ln\left(\frac{H_t}{L_t}\right)}{\sqrt{P_t \cdot VOL_t}}, \tag{20}$$

where H_t is the high price in day *t*, L_t is the low price in day *t*.

The adjusted close spread is calculated as in Chung and Zhang (2014), where bid and ask prices are replaced by the high and the low prices, respectively. This reason for this adjustment is that bid and ask data are not publicly offered, but could be replaced with the available high and low prices. The high-low range HLR, as calculated in the following way:

$$HLR_t = \frac{H_t - L_t}{0.5(H_t + L_t)}.$$

For our sample of stocks we estimate multifactor models with liquidity proxy as an additional variable in a form:

$$\begin{aligned} R_{it} - r_{Ft} &= b_{i0} + b_{i1}\left(R_{Mt} - r_{Ft}\right) + b_{i2}SMB_t + b_{i3}HML_t + b_{i4}MOM_t + \\ &+ b_{i5}LIQ_t + e_{it}, \end{aligned} \tag{21}$$

where LIQ_t is represented either by $ILLIQ, VoV$ or HLR.

In Table 7 we show the estimates of the models only for these stocks, for which liquidity premium in multifactor models is significant. As our measures of liquidity are in fact illiquidity proxies, we expect the parameters to have negative signs. It's true in all but one case – we got a positive value of the parameter for PEO. In other cases we may conclude that the more liquid the stock is, the higher is the return: there exists positive premium for liquidity.

Table 7. The estimates of multifactor models with liquidity premium

	KGH	KGH	KGH	AGO	MBK	PEO	PKN
b_{i0}	0.02	0.02	0.02	–0.01	**0.03****	0.00	**0.04***
b_{i1} (Market)	**1.26*****	**1.26*****	**1.26*****	**1.22*****	**1.29*****	**1.07*****	**0.87*****
b_{i2} (SMB)	0.21	0.21	0.21	–0.13	0.16	**–0.22**	–0.12
b_{i3} (HML)	**0.54*****	**0.54*****	**0.52*****	–0.03	**0.19***	0.04	**0.24**
b_{i4} (MOM)	0.04	0.04		**0.07**			
b_{i5} (ILLIQ)	**–6.47***		**–6.35***			**2.70***	
b_{i5} (VoV)		**–3.86***					
b_{i5} (HLR)					**–0.82***		**–1.14****
R^2	0.48	0.48	0.48	0.41	0.53	0.63	0.51

Note: The parameters presented in the Table are from multifactor models with liquidity premiums, proxied by either ILLIQ, VoV or HLR. The stars *, **, *** denote statistical significance at α level 0.1, 0.05 and 0.01 respectively.

Problems and solutions

Problem I

Let us consider two stocks, ACP and KGH:

Stock	Beta	Standard deviation of residuals
ACP	0.60	5.04
KGH	1.32	8.33

The risk of a market portfolio measured by the standard deviation is equal to 3.84.

1. Assume that the one-factor market model's assumptions hold what is the risk measured by standard deviation of portfolio consisting of 70% ACP?
2. Assume that one-factor model does not hold and the correlation of errors is 0.5. Calculate the risk of a portfolio measured by standard deviation.
3. Consider ACP and KGH in a different setting, assuming that 2-factor model holds in which GDP changes are the second factor.

Stock	Market beta	GDP beta	Standard deviation of residuals
ACP	0.6	1.1	5.04
KGH	1.32	0.4	8.33

The standard deviation of the index representing changes in GDP is 0.25. There is no autocorrelation in the residuals from the model. Again, what is the risk of a portfolio measured by standard deviation?

Solution

1. The risk of the portfolio measured with variance of returns is calculated in the following formula (on a similar basis as eq. (9)):

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma_{e_p}^2.$$

Thus, we need portfolio beta, β_P, and the variance of the residuals. Based on information that the one-factor assumptions hold, we may use the following formula:

$$\sigma_{e_p}^2 = \text{var}\left(e_P\right) = \sum_{i=1}^{n} x_i^2 \sigma_{e_i}^2.$$

So:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma_{e_p}^2 = (0.7 \cdot 0.6 + (1-0.7) \cdot 1.32) \hat{\ } 2 \cdot 3.84^2 + \left(0.7 \cdot 5.04\right)^2 +$$
$$+\left(0.3 \cdot 8.33\right)^2 = 28.51$$

where:

$$\beta_P = x_{ACP} \cdot \beta_{ACP} + x_{KGH} \cdot \beta_{KGH} = 0.7 \cdot 0.6 + (1-0.7) \cdot 1.32 = 0.816,$$

$$\sigma_M^2 = 14.7456 = 3.84^2,$$

$$\text{var}\left(e_P\right) = x_{ACP}^2 \cdot \sigma_{e,\, ACP}^2 + x_{KGH}^2 \cdot \sigma_{e,\, KGH}^2 = 0.7^2 \cdot 5.04^2 + 0.3^2 \cdot 8.33^2 = 18.69.$$

Thus the risk of the portfolio measured by the standard deviation is $\sqrt{28.51} = 5.34$.

2. If the one-factor model does not hold, then we should consider the correlation of the errors in the formula for $\text{var}(e_P)$:

$$\text{var}(e_P) = \mathbf{x}'\mathbf{V_e x},$$

where:

$$\mathbf{V_e} = \begin{pmatrix} 5.04^2 & 20.99 \\ 20.99. & 8.33^2 \end{pmatrix}, \quad \mathbf{x}' = [0.7\,0.3],$$

and $20.99 = \sigma_{e,ACP} \cdot \sigma_{e,KGH} \cdot \rho = 5.04 \cdot 8.33 \cdot 0.5.$

The variance of the return is calculated as:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma_{e_p}^2 = (0.7 \cdot 0.6 + (1 - 0.7) \cdot 1.32)^2 \cdot 3{,}84^2 + (0.7 \cdot 5.04)^2 +$$
$$+ (0.3 \cdot 8.33)^2 + 2 \cdot 0.7 \cdot 0.3 \cdot 20.99 = 37.33.$$

3. In a two-factor model the variance of the portfolio is given as:

$$\sigma_P^2 = \beta_{P,M}^2 \sigma_M^2 + \beta_{P,GDP}^2 \sigma_{GDP}^2 + \sigma_{e_p}^2.$$

Thus:

$$\sigma_P^2 = (0.7 \cdot 0.6 + 0.3 \cdot 1.32)^2 \cdot 3{,}84^2 + (0.7 \cdot 1.1 + 0.3 \cdot 0.4)^2 \cdot 0.25^2 +$$
$$+ (0.7 \cdot 5.04)^2 + (0.3 \cdot 8.33)^2 = 28.56$$

The risk of the portfolio measured by the standard deviation is $\sqrt{28.56} = 5.34.$

Problem II

Suppose you are considering the purchase of shares of one of two banks. As a result of the Fama-French three factor model on monthly returns for the past five years, you obtain the following three factors coefficient estimates:

	Bank 1	**Bank 2**
Market factor	0.83	1.05
SMB factor	0.02	–0.19
HML factor	0.20	0.03

Describe what types of stocks are Bank1 and Bank 2 on the basis of data given?

Solution

Bank1 seems to be a defensive stock with market beta coefficient lower than 1. As SMB coefficient is positive, it might be a small company that behaves as an income stock (positive HML coefficient). Bank2 is an aggressive stock, with-market beta higher than 1), it behaves as a big stock (SMB factor coefficient is negative) and represents a kind of income stock (HML factor coefficient is positive, but not significantly different from zero.

Problem III

Consider the data for two factors (1 and 2) and two stocks (A and B):

$$f_0 = 0.22,\ \ f_1 = 0.05 \ \text{ and } \ f_2 = -0.18,\ \ b_{1A} = 1.46,\ \ b_{1B} = 1.06,\ \ b_{2A} = 1.20 \ \text{ and } \ b_{2B} = -0.19.$$

a. Compute the expected returns for both stocks.
b. Assuming that stock A is currently priced at 23 EUR, while stock B is currently priced at 16 EUR and both stocks are expected to pay dividend of 1 EUR during the coming year. What is the expected price for each security one year from now based on the given model?

Solution

The expected returns:

$$E(R_i) = f_0 + f_1 b_{1i} + f_2 b_{2i},$$

$$E(R_A) = 0.22 + 0.05 \cdot 1.46 + (-0.18) \cdot 1.2 = 0.077,$$

$$E(R_B) = 0.22 + 0.05 \cdot 1.06 + (-0.18) \cdot (-0.19) = 0.287.$$

The expected price of the securities:

$$E(R_t) = \frac{P_t - P_{t-1} + D_t}{P_{t-1}},$$

$$P_t = E(R_t) P_{t-1} - D_t + P_{t-1},$$

$$P_{t,A} = 0.077 \cdot 23 - 1 + 23 = 23.77,$$

$$P_{t,B} = 0.287 \cdot 16 - 1 + 16 = 19.59.$$

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CHAPTER 2
FINANCIAL RISK MEASUREMENT

2.1. Financial risk management

Financial risk is an inherent factor which firms must accept in their economic activity. The majority of firm managers are aware that proper risk management is not only a way to increase their economic efficiency, but also ensures the existence on a market in general. There are many ways to manage the risk depending on type of risk, source of risk or target the manager wants to achieve and tools he or she is going to take advantage of. Some risks can be directly managed and other risks are moved beyond the control of company management.

In our day-to-day life, the risk and the uncertainty are treated as synonyms but both terms do not mean exactly the same. The formal distinction comes from Knight (1921). A risk can be understood as a danger that some events will cause an undesirable outcome in the financial situation of the firm or the investor. It is possible to identify its future alternatives, assuming that the chances (probabilities) of possible alternatives are known. The term *uncertainty* is defined as a state in which the possible future alternatives and chances of their occurrence are unknown. The difference between a risk and uncertainty can be expressed as follows:

1) a risk is defined as the situation of losing something worthy and future possible events are known; we face uncertainty if we have no knowledge about the future events,
2) in case of a risk, probabilities of possible outcomes are known, and it is not possible to define probabilities in case of uncertainty,
3) a risk, conversely to uncertainty, can be measured through theoretical models whereas uncertainty cannot be measured in quantitative ways,
4) a risk can be managed and minimized by many ways, but uncertainty is beyond the control of an investor.

There are three main sources of financial risks:

1) financial risks arising from a firm's exposure to changes in market prices, such as interest rates, exchange rates, stock and indices or commodity prices–this is a **market risk**,

2) financial risks arising from the actions of other organizations such as sellers, customers, and counterparties in transactions–this is a **credit risk**,
3) financial risks resulting from internal actions or failures of the firm, people, processes, and systems–this is an **operational risk**.

Many times all types of risk are closely tied together. The four examples below describe famous bankruptcy cases and identify the type of risk which led to bankruptcy.

Example. Barings Bank Ltd. was a British merchant bank based in London (the world's second oldest one). Twenty-something Nicholas Leeson was a market trader and back-office manager of the bank in a Singapore department. Initially, he transferred the bank's internal orders for execution, and then he promoted and managed a team of traders. Leeson speculated in more and more capital. He invested in futures contracts for the Japanese stock market index Nikkei. He took long positions although the market was falling (a market risk). For a long time he was able to conceal the losses (an operational risk) from the London headquarters of the bank. Unfortunately, after an earthquake in Kobe on January 17, 1995, the index dropped below 18 thousand points and Leeson no longer had the money for margin, and his losses on investment exceeded 800 million pounds. It was twice as high as the capital of the bank. The bank was unable to handle such a large loss and had to declare bankruptcy (a credit risk). Finally, Nick Leeson was arrested at the Frankfurt airport and accused of fraud. He was sentenced to 6.5 years in prison.

Example. Long Term Capital Management (LTCM)–an investment fund was created in 1993 by three genius people: Nobel Prize winners Robert Merton and Myron Scholes and the legend of bond market John Meriwether. The fund focused on arbitrage and speculation transactions burdened with high financial leverage. Its key concept was as follows: the reduction of fund's volatility through hedging (it involves the use of derivatives in investment strategies) will allow to increase the scale of planned transactions with the same level of volatility as observed in the case of unhedged transactions, but with a higher expected return. In 1995 the fund realized a return of 63%, and in 1996–57%. In 1997 LTCM invested in Russian bonds and hedged its position in ruble by forward contracts. On August 17, Russia announced the suspension in payment of its obligations (a credit risk) and the devaluation of the ruble. The regulation allowing national banks to disregard currency contracts for a month was introduced. As a result of the Russian crisis and bankruptcy of Russia, the fund suffered a loss of $4.6 billion.

Example. Enron Corporation was an American energy, commodities and services company (a US blue chip) based in Texas. At the end of 2001 it was revealed that its financial reports were systematically falsified using creative accounting (an operational risk). This action was conducted in cooperation with Enron's auditor Arthur Andersen. Many accusations were made, connected with the falsification of financial statements, frauds related to securities, the use of confidential information, as well as making false statements. Finally Enron went bankrupt in November 2004. Employees were fired and two CEOs were brought to justice. Arthur Andersen was accused in the process of the collapse of Enron company for help in hiding company debts and falsifying financial statements of Enron and went bankrupt in 2002.

Example. Lehman Brothers Holding Inc. was a global financial services firm, the fourth-largest investment bank in the United States (behind Goldman Sachs, Morgan Stanley, and Merrill Lynch). The company existed for almost 160 years. Its collapse on September 15, 2008 was so far the largest in the history of finance and it has been considered as the symbolic beginning of the global financial crisis. Mass mortgage lending to people who did not have creditworthiness was common practice in the US in 2004-2005. The increasing problem with their repayments exploded in autumn 2008 when interest rates increased from 1% to over 5%. Losses on such loans (a credit risk) and, consequently, depreciation of shares (a market risk) were the main reason for the bank's collapse. The bank's collapse caused panic on the stock exchanges. The American part of Lehman Brothers' assets was overtaken by Barclays and the European and Asian one by the Japanese Nomura bank.

We should ask then and answer the following question: was it possible to avoid the said bankruptcies by managing the risk? The answer is yes, but behind all those examples lie human decisions, driven by lust for profit and fame and inflated self-confidence. Isaac Newton said: "I can calculate the motion of heavenly bodies, but not the madness of people."

Apart from the three main types of financial risk we can we also define (Hull, 2010):

1) a liquidity risk which appears when an organization has a problem with conversion of its assets into cash. It can be understood in two ways. In investing terms, investors face a liquidity risk based on the likelihood that they may be forced to sell a security below its market value. In economics and business management, liquidity refers to the ability of a firm to ensure continuity of settlement of its liabilities,
2) a legal risk which is the risk of disadvantageous changes in legal regulations,

3) an events risk which appears according to a singular event like terrorist attacks or natural disasters,
4) a model risk which is the risk of an invalid model of the risk or pricing.

Financial risk management is a process to deal with the risk resulting from financial markets. It involves assessing the financial risks facing an organization and developing management strategies consistent with internal priorities and policies. Strategies for risk management often involve derivatives.

The process of financial risk management can be summarized as follows:
1) identify financial risks factors,
2) measure the risk level and determine an appropriate level of risk tolerance,
3) implement risk management strategy in accordance with policy,
4) report, monitor, and refine as needed.

The risk management process involves both internal and external analysis. This part of the process involves identifying and prioritizing the financial risks faced by an organization and understanding their relevance. There are three broad alternatives for managing risks:
1) do nothing and accept all risks,
2) hedge a portion of exposures by determining which exposures can and should be hedged,
3) hedge all exposures possible.

2.2. Market risk measurement

Risk measurement attempts to quantify the risk of losses due movements in financial market variables. The variables include: interest rates, foreign exchange rates, equities, commodities. Positions can include cash or derivative instruments.

Desirable properties of risk measures – coherent risk measure ρ (Artzner, Delbaen, Eber, & Heath, 1999):

Let X be a random variable that represents the net worth of the position at the end of the trading period. The value $\rho(X)$ is a measure of risk. We require that the measure ρ should have the following properties.

1. Monotonicity: $X_1 \le X_2 \Rightarrow \rho(X_1) \ge \rho(X_2)$.

If portfolio has systematically lower value then another, it must have a greater risk.

2. Translation invariance: $\rho(X+k) = \rho(X) - k$.

Adding cash k to the portfolio reduces the risk of portfolio by k. The lowest value of portfolio is reduced.

3. Homogeneity: $\rho(bX)=b\rho(X)$.

Increasing the size of a portfolio by a factor b should scale its measure by the same factor b.

4. Subadditivity: $\rho(X_1+X_2)\le\rho(X_1)+\rho(X_2)$.

The risk of a portfolio must be less than the sum of separate asset risks. This is a diversification effect.

There are three groups of risk measures:

1) volatility measures–measure deviations of variable values (e.g. standard deviation, variance),
2) sensitivity measures–measure sensitivity of the investment to the risk factors (e.g. duration, CAPM beta, option Greeks),
3) downside risk measures–measure losses on investment (e.g. semi-standard deviation, Value at Risk, Expected Shortfall).

Financial risk management process focuses on downside risk measures, especially on Value at Risk and Expected Shortfall. The measures are generally used by financial institutions to determine the potential losses in their institutional portfolios. Value at Risk is a measure of risk that describes the risk in probability terms. It is widely used by all the financial institutions, banks, investment funds and corporations. The advantage of VaR is that it describes the total risk in a portfolio of financial assets by only one value. VaR indicates how big the maximum loss over target horizon is such that there is a low, pre-specified probability that the actual loss will be equal or larger. The general formula is as follows:

$$\Pr(P\le P_0-\mathrm{VaR})=\alpha,$$

where:

P_0 – initial value of portfolio,
P – final value of portfolio (random variable),
α – tolerance level (small value).

VaR says how bad things can get. In other words we can interpret VaR in such a way that there is a probability c that our portfolio will not lose more than VaR in the next few days.

$$\Pr(P_0-P<\mathrm{VaR})=c,$$

where:

$X=P_0-P$ – portfolio loss,
$c=1-\alpha$ – confidence level (value close to 1).

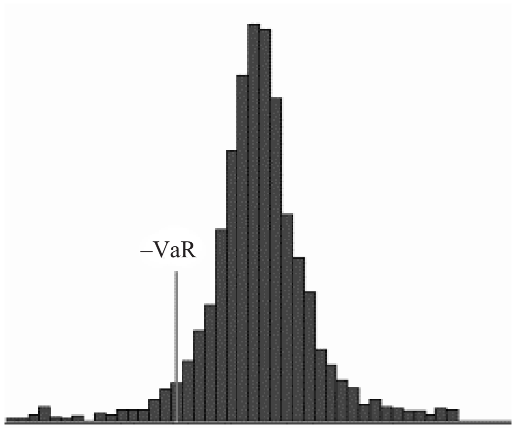

Figure 1. Histogram of portfolio changes over a given horizon

Example. Consider the portfolio worth \$100 million. 10-day Value at Risk with a 5% level of tolerance is equal to \$7 million. We are 95% certain that our portfolio will not lose more than \$7 million in the next 10 days. In other words, there is only 5% chance that our portfolio will lose \$7 million or more in the next 10 days.

To describe VaR two parameters are needed: confidence (or tolerance) level and time horizon (Jorion, 2011).

1. Confidence level c
   - the higher the confidence level, the greater the VaR value,
   - the higher the confidence level, the fewer exceedances of VaR and poor statistical properties of the measure.
2. Horizon T
   - the longer the horizon, the greater the VaR measure,
   - VaR can be extended from 1-day horizon to T-days by using squared-root of time rule:

$$\text{VaR}(T - day) = \text{VaR}(1 - day)\sqrt{T}.$$

This adjustment is used under i.i.d. (independent and identical distributed) assumption for returns that have a normal distribution with zero mean.

The easiest way to measure VaR is to use the following expression:

$$\text{VaR}_c(X) = q_c \cdot SD(X),$$

where:

$SD(X)$ – standard deviation of losses,

q_c – c-th quantile of standardized normal distribution,
If $c = 0.95$ then $q_c = 1.645$ and $\text{VaR}_c(X) = 1.645 \cdot SD(X)$,
If $c = 0.99$ then $q_c = 2.326$ and $\text{VaR}_c(X) = 2.326 \cdot SD(X)$.

Disadvantages of VaR

1. VaR does not indicate how much an investor can lose when VaR is exceeded.
2. VaR is measured with some error, and different statistical methodologies can lead to different VaR numbers.
3. VaR is not a coherent risk measure generally; VaR satisfies the subadditivity property in the case of elliptical distribution (i.e. normal distribution).

The Basel Committee Rules for VaR parameters

The Basel committee is a powerful group of bank regulators that meets regularly to agree risk management rules that affect every bank in the world. The most important rule is the “capital adequacy ratio”, which sets the minimum reserve of capital a bank must keep to absorb losses on their loans. Committee members come from Argentina, Australia, Belgium, Brazil, Canada, China, France, Germany, Hong Kong SAR, India, Indonesia, Italy, Japan, Korea, Luxembourg, Mexico, the Netherlands, Russia, Saudi Arabia, Singapore, South Africa, Spain, Sweden, Switzerland, Turkey, the United Kingdom and the United States. The Committee’s Secretariat is located at the Bank for International Settlements (BIS) in Basel, Switzerland. Basel requirements due to VaR calculation in Internal Model Approach are as follows:

1. A time horizon of 10 trading days, or two calendar weeks.
2. A 99-percent confidence level.
3. An observation period based on at least a year of historical data and updated at least once a quarter (amendment introduced in 2009, require updating at least once a month).

The Market Risk Charge (MRC) is measured as follows:

$$MRC_t = \max\left(k\frac{1}{60}\sum_{i=1}^{60}\text{VaR}_{t-i}, \text{VaR}_{t-1}\right) + SRC_t,$$

where:

k – multiplier no less than 3, depending on the number of VaR exceedances in the backtesting procedure,

SRC – the specific risk charge related to the individual security.

A bank experiencing an excessive number of violations will be penalized by having to set aside a larger amount of risk capital.

Example. Calculate Value at Risk according to Basel rules, knowing that VaR (95%, 1-day) = \$30 000.

We assume a normal model for VaR, thus for daily VaR we have:

$$\mathrm{VaR}_{0.95} = 1.645 \cdot SD$$

$$\mathrm{VaR}_{0.99} = 2.326 \cdot SD = \frac{2.326}{1.645}\mathrm{VaR}_{0,95},$$

then 10-day VaR:

$$\mathrm{VaR}_{0.99}(10 - day) = \frac{2.326}{1.645} \cdot \sqrt{10} \cdot \mathrm{VaR}_{0.95} = 134{,}142.1.$$

Stress-testing

Stress-testing is a key risk management step. Its goal is to identify of potential vulnerability. Stress-testing is analyzed by scenario analysis–it exhibits the portfolio to large movements in financial market variables. The scenarios can be created by:

1) moving key variables at the time,
2) using historical scenarios,
3) creating prospective scenarios.

Expected Shortfall (conditional VaR, tail conditional expectation, conditional loss, expected tail loss)–tells us how much one can lose on average if the losses exceed the VaR

$$\mathrm{ES} = E[X \mid X \geq \mathrm{VaR}],$$

where X is a portfolio or an asset loss. It is a measure that is used in risk measurement process as a supplement of VaR.

2.2.1. Value at risk methods

Financial rates, exchange rates and prices are affected by a number of factors. These are general economic conditions, government debt and policy, financial and political stability, relative strength of currencies, inflation, monetary policy of the central bank, investors speculations and many others. It is essential to understand how much the factors impact markets and potential risks of an organization. On the other hand, it is impossible to capture the huge amount of complexity in risk models. Risk measurement is a trade-off between accuracy and usefulness. There are three methods of calculating VAR: the historical simulation method, the variance-covariance method and Monte Carlo simulation. We present them for univariate and multivariate case.

2.2.1.1. Historical simulation method (HS)

Historical simulation is a nonparametric method and the simplest way for estimating risks. It takes into account an empirical distribution.

Univariate case. We observe data from 1 to t, the current asset value is P_t. Then we calculate daily returns R_i and generate tomorrow's prices $SP_i = (1+R_i)P_t$, for $i = 2, \dots, t$. Then we calculate changes of simulated asset $\Delta SP_i = SP_i - P_t = R_i P_t$, $i = 2, \dots, t$. Losses are on the left-hand side of the distribution. Finally, we calculate VaR as a α-th quantile of changes in value.

Multivariate case. We observe data from 1 to t for each asset. The current portfolio value is P_t. We generate $i = 2, \dots, t$ scenarios for tomorrow $SP_{i,j} = (1+R_{i,j})P_{t,j}$ for $j=1, \dots, N$ components of portfolio. Generated tomorrow's portfolio values are $SP_i = \sum_{j=1}^{T} SP_{i,j}$. Then the procedure is like in the univariate case, we calculate changes of simulated portfolio $\Delta SP_i = SP_i - P_t$, $i=2, \dots, t$. Again, losses are on the left-hand side of the distribution, and finally we calculate VaR as a α-th quantile of changes in value.

Advantages (A) and disadvantages (D) of HS

- we do not assume a theoretical distribution class (A),
- the method can be easily implemented to the assets portfolio (A),
- all past scenarios have the same influence on tomorrow's VaR (D),
- we cannot use a square root of time rule to calculate a T-day VaR (D).

ES estimation

ES is calculated by taking the average of all exceedances of VaR.

Example. Consider a portfolio of three stocks. Portfolio consists of 100 stocks 1, 200 stocks 2, and 300 stocks 3. We calculate VaR of separate position in stocks and of the whole portfolio. The first seven rows and the last one are presented in Table 1.

Table 1. Calculation of VaR for separate positions

i	$P_{i,1}$	$P_{i,2}$	$P_{i,3}$	$R_{i,1}$	$R_{i,2}$	$R_{i,3}$	$SP_{i,1}$	$SP_{i,2}$	$SP_{i,3}$	$\Delta SP_{i,1}$	$\Delta SP_{i,2}$	$\Delta SP_{i,3}$
1	25.11	10.63	4.12									
2	25.50	10.68	4.36	1.55%	0.47%	5.83%	2655.53	1965.19	1238.21	40.53	9.19	68.21
3	25.06	10.56	4.35	–1.73%	–1.12%	–0.23%	2569.76	1934.09	1167.31	–45.24	–21.91	–2.69
4	26.20	10.58	4.31	4.55%	0.19%	–0.92%	2733.98	1959.72	1159.24	118.98	3.72	–10.76
5	26.99	10.59	4.30	3.02%	0.09%	–0.23%	2693.97	1957.76	1167.31	78.97	1.76	–2.69
6	26.75	10.39	4.18	–0.89%	–1.89%	–2.79%	2591.73	1919.03	1137.36	–23.27	–36.97	–32.64
7	26.33	10.27	4.10	–1.57%	–1.15%	–1.91%	2573.94	1933.51	1147.65	–41.06	–22.49	–22.35
…	…	…	…	…	…	…	…	…	…	…	…	…
t	**26.15**	**9.78**	**3.90**	1.55%	0.31%	–0.26%	2655.53	1962.06	1166.96	40.53	6.06	–3.04

Note: Generated tomorrow's values are multiplied by position in the individual stocks, 100, 200 and 300 respectively.

Table 2. Calculation of VaR for the portfolio

i	$P_{i,1}$	$P_{i,2}$	$P_{i,3}$	$R_{i,1}$	$R_{i,2}$	$R_{i,3}$	$SP_{i,1}$	$SP_{i,2}$	$SP_{i,3}$	SP_i	ΔSP_i
1	25.11	10.63	4.12								
2	25.50	10.68	4.36	1.55%	0.47%	5.83%	2655.53	1965.19	1238.21	5858.93	117.93
3	25.06	10.56	4.35	–1.73%	–1.12%	–0.23%	2569.76	1934.09	1167.31	5671.16	–69.84
4	26.20	10.58	4.31	4.55%	0.19%	–0.92%	2733.98	1959.72	1159.24	5852.94	111.94
5	26.99	10.59	4.30	3.02%	0.09%	–0.23%	2693.97	1957.76	1167.31	5819.04	78.04
6	26.75	10.39	4.18	–0.89%	–1.89%	–2.79%	2591.73	1919.03	1137.36	5648.12	–92.88
7	26.33	10.27	4.10	–1.57%	–1.15%	–1.91%	2573.94	1933.51	1147.65	5655.10	–85.90
…	…	…	…	…	…	…	…	…	…	…	…
t	**26.15**	**9.78**	**3.90**	1.55%	0.31%	–0.26%	2655.53	1962.06	1166.96	5784.55	43.55

VaR for separate position is a minus quantile of elements in data vector in the last three columns. We get 53.23, 77.30 and 46.67.

To calculate VaR for whole portfolio, we have to first summarize $SP_{i,j}$. Then we subtract current portfolio value, here $26.15 \cdot 100 + 9.78 \cdot 200 + 3.90 \cdot 300 = 5741$ and obtain changes of portfolio to calculate its quantile.

VaR is a minus quantile of values in the lost column. We get 155.12.

2.2.1.2. Variance-covariance method (VC)

Variance-covariance method is a parametric method. The most popular is the model with normal distribution for returns assuming zero mean. Both VaR and ES have closed form solutions (Jorion, 2005).

Univariate case

$$\text{VaR}_\alpha = -\sigma q_\alpha P_t,$$

where:

P_t – asset value at time t,

σ – standard deviation of returns,

q_α – α -th quantile of returns distribution $(q_{1\%} = -2.326,\ q_{5\%} = -1.645)$.

Multivariate case

$$\text{VaR}_\alpha = -\sigma q_\alpha P_t,$$

where:

P_t – portfolio value at time t,

and variance of portfolio is as follows:

$$\sigma^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w},$$

$\mathbf{\Sigma}$ – covariance matrix of returns,

$\mathbf{w}$ – vector of portfolio weights.

Advantages (A) and disadvantages (D) of VC method

- a method easy to calculate (A),
- we have to assume a theoretical distribution class, usually normal distribution (D),
- the method can be easy implemented to the assets portfolio in case of normal distribution (A),
- all past scenarios have the same influence on tomorrow's VaR (D),

– we can use a square root of time rule to calculate a *T*-day VaR in case of normal distribution (A),
– too thin tails of normal distribution which cause underestimation of the risk (D).

ES estimation

$$ES = -P_t \frac{\sigma\phi(q_\alpha)}{\alpha},$$

where:

P_t – value of asset or portfolio at time t,
ϕ – probability density function of the standard normal distribution.

2.2.1.3. Monte Carlo methods (MC)

Monte Carlo is a simulation method for risk measurement. Usually MC is based on geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma \in \sqrt{dt}$$

where:

μ – mean,
σ – volatility,
dt – time interval,
$\in$ – standard normal random variable.

Using Itô lemma we receive the following asset value:

$$P_t = P_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \in \sqrt{t}}.$$

Univariate case. We generate many (e.g. 10 000) realizations of $\in$, and according to the upper formula, we calculate the values of an asset after t time. Next we calculate the changes of asset $\Delta P_t = P_t - P_0$ and finally α -th quantile as VaR.

Multivariate case. Because individual assets are correlated, random variables z_i should be generated in a way to have correlated variables with the correlation matrix **R**. The method of solving this problem is called the Cholesky decomposition. Correlated random variables have the form:

$$\begin{bmatrix} z_1 \\ \dots \\ z_L \end{bmatrix} = \mathbf{T} \begin{bmatrix} \in_1 \\ \dots \\ \in_L \end{bmatrix},$$

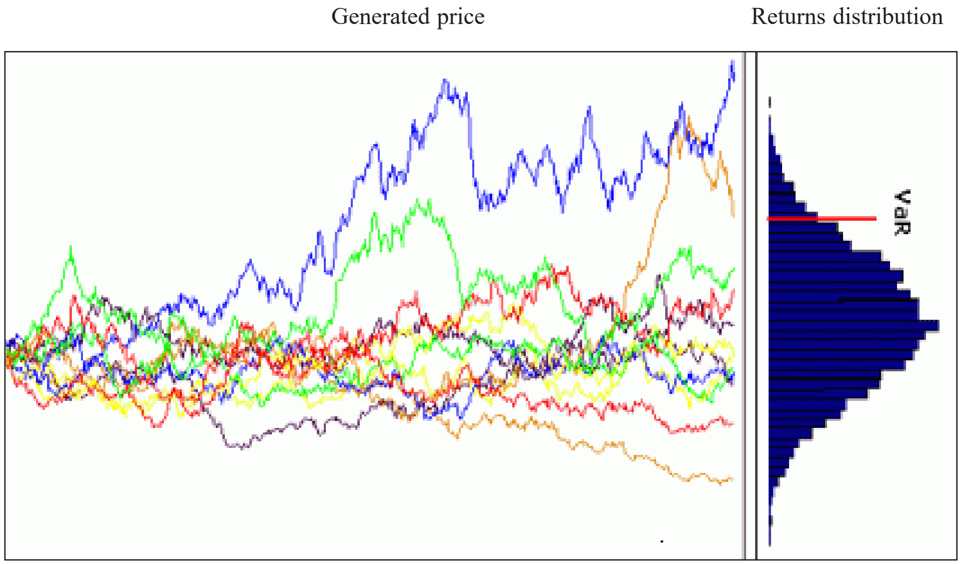

Figure 2. Simulations of prices and returns

where:

ϵ_i – uncorrelated random variables,

$\mathbf{T}$ – lower triangular matrix (zeros above the diagonal) fulfilling:

$\mathbf{R} = \mathbf{TT}'$.

Example. Find a closed form solution for two correlated assets with a correlation between returns equal to ρ.

Matrix $\mathbf{T}$ has the form:

$$\mathbf{T} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}.$$

and can be decomposed into Cholesky factors

$$\mathbf{R} = \mathbf{TT}' \Rightarrow \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix}.$$

After simplification we obtain:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{21}a_{11} \\ a_{21}a_{11} & a_{21}^2 + a_{22}^2 \end{bmatrix},$$

and hence

$$\begin{cases} a_{11} = 1 \\ a_{11}a_{21} = \rho \\ a_{21}^2 + a_{22}^2 = 1. \end{cases}$$

Solving the system of equations we have: $a_{11} = 1$, $a_{21} = \rho$, $a_{22} = \sqrt{1-\rho^2}$. Finally, we obtain correlated random variables:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \rho\epsilon_1 + \sqrt{1-\rho^2}\epsilon_2 \end{bmatrix}.$$

The prices after time t are equal to:

$$P_1(t) = P_1(0)e^{\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1\epsilon_1\sqrt{t}},$$

$$P_2(t) = P_2(0)e^{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2(\rho\epsilon_1 + \sqrt{1-\rho^2}\epsilon_2)\sqrt{t}}.$$

Advantages (A) and disadvantages (D) of MC methods

- flexibility, we can use various stochastic processes (A),
- it allows to take into account various factors of risk (A)
- all past scenarios have the same influence on tomorrow's VaR (D),
- the method is complex and it is not easily implemented in portfolio case (D),

Value at Risk measure can be easily applicable for measuring nonlinear instruments risks, like bonds or options.

VaR for bonds. Let MD be a modified duration and y be a yield of bond. Using duration approximation we have

$$\Delta P = -MD \cdot P \cdot \Delta y,$$

and

$$\sigma(\Delta P) = MD \cdot P \cdot \sigma(\Delta y).$$

The Risk Metrics[1] uses constant $\sigma(\Delta y/y)$, hence volatility of yield changes is as follows:

$$\sigma(\Delta P) = y \cdot \sigma\left(\frac{\Delta y}{y}\right).$$

Finally:

$$\sigma(\Delta P) = MD \cdot P \cdot y \cdot \sigma\left(\frac{\Delta y}{y}\right),$$

and VaR in "normal world" is:

$$\text{VaR}_\alpha = -MD \cdot P \cdot y \cdot \sigma\left(\frac{\Delta y}{y}\right) \cdot q_\alpha.$$

Example. Investor has a position in bond with face value of \$1 million. The bond price is 97.5, accrued 2.15%, annual yield 5.42%, duration 9.15 and yield volatility 12%, assume 250 trading days a year. Calculate 10-day VaR at the 99% confidence level for his position.

$$P = \$1 \text{ million} \cdot \frac{(97.5 + 2.15)}{100} = \$0.9965 \text{ million}$$

$$MD = \frac{D}{1+y} = \frac{9.15}{1+0.0542} = 8.6796.$$

Because we have annual volatility, we have to divide it by $\sqrt{25}$ to obtain 10--day volatility, thus

$$\text{VaR} = 8.6796 \cdot \$0.9965 \text{ million} \cdot 0.0542 \cdot 0.12 / \sqrt{25} \cdot 2.326 = \$26{,}169.61.$$

VaR for options. Let Δ be the option delta, and P be the option price, S be underlying asset price, delta approximation is:

$$dP = \Delta \cdot dS.$$

[1] RiskMetrics is a methodology that contains techniques and data sets used to calculate the VaR of a portfolio of investments. RiskMetrics was launched in 1994, as a technical document in October 1994 by J. P. Morgan. It is widely available for practitioners and the general public.

VaR can be calculated as follows:

$$VaR_\alpha = |\Delta| \cdot VaR(dS) = -|\Delta| \cdot S \cdot \sigma\left(\frac{dS}{S}\right) \cdot q_\alpha .$$

The absolute value is used because the put option delta is negative. The accuracy of the approximation depends on the model that is used to option valuation and model risk appears in this case. The most popular option pricing model is the Black-Scholes model that gives delta in closed-form solution (see: Hull, 2015).

Example. At-the-money put option on Polish WIG20 stock index is struck on 24 000 PLN. An annual volatility of index is 15%. What is 10-day VaR at the 95% confidence level? Assume 250 trading days per year and zero mean normal distribution.

The delta of the ATM put must be around –0.5, which implies:

$$\text{VaR} = 0.5 \cdot 24{,}000 \cdot 0.15 / \sqrt{25} \cdot 1.645 = 592.2\, PLN.$$

2.2.2. Backtesting

Backtesting is a procedure that is used to compare various models of risk. It allows to confirm if the risk model measures the risk correctly. It aims to take an *ex ante* Value at Risk forecasts from a particular model and compare them with *ex post* realized returns. The most popular application in research is Kupiec's test.

Kupiec's Proportion of Failures Test

The most important test for the number of exceedances. Define α^* as the actual frequency of VaR exceedances. The test considers the following hypotheses:

H_0: $\alpha = \alpha^*$

H_1: $\alpha \neq \alpha^*$.

Test statistics is of the form:

$$LR = 2\left(\ln\left(\left(\frac{N}{n}\right)^N \left(1-\frac{N}{n}\right)^{n-N}\right) - \ln\left(\alpha^N (1-\alpha)^{n-N}\right)\right),$$

where:

n – sample size,

N – number of VaR exceedances.

$LR \sim \chi^2$ (1) when H_0 is true.

The Basel Committee's Traffic Light Coverage Test

The test is the Basel Committee's procedure based on no statistical theory for hypothesis testing. It is important because of its wide use by banks. Back-tests is to be performed quarterly using the most recent 250 days of data. Based on the number of exceedances experienced during that period, the VaR measure would be categorized as falling into one of three colored zones, as it is presented in Table 3.

Table 3. Color zones in traffic light coverage test

Zone	Number of exceedances	Multiplier	Cumulative probability for c = 0.99
Green	0	3	0.0811
	1	3	0.2858
	2	3	0.5432
	3	3	0.7581
	4	3	0.8922
Yellow	5	3.4	0.9588
	6	3.5	0.9863
	7	3.65	0.9960
	8	3.75	0.9989
	9	3.85	0.9997
Red	10 or more	4	0.9999

Cumulative probability is calculated with a binomial distribution, assuming 0,99 probability that exceedance of VaR does not occur. A bank experiencing an excessive number of violations is penalized by higher multiplier and, as consequence, it has to set aside a larger amount of risk capital to cover greater market risk. Moreover, if the number of exceedances is in the red zone, banks will have to take an immediate action to reduce their risk or improve the VaR model. Otherwise, they may even lose their banking license.

Example. A risk manager tested VaR model at 95% confidence level. VaR forecasts were calculated on the basis of 250 last trading days and were daily updated in the period of 07.01.2009–29.12.2017. In the period he got 2263 returns and VaR forecasts which were compared. He received 91 exceptions. Figure 3 shows returns and VaRs in the period. Is his model correct?

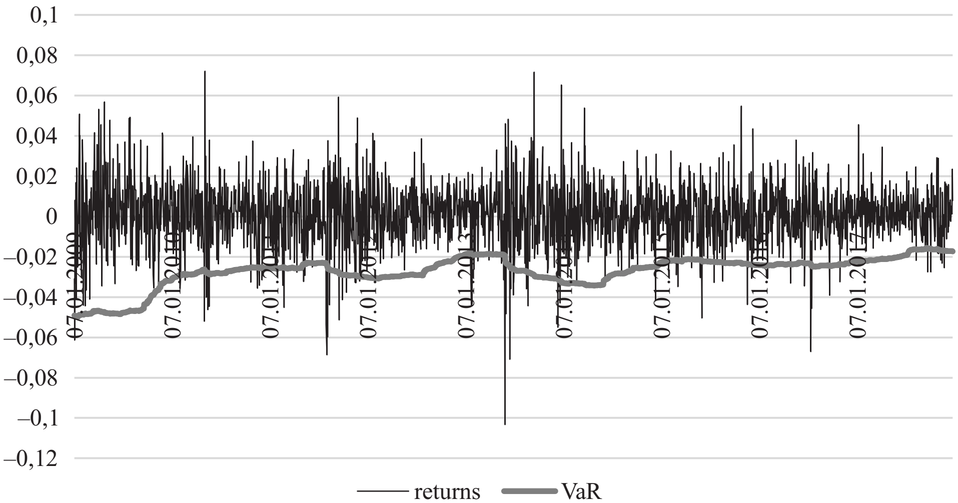

Figure 3. Returns and VaR

Taking Traffic Light Coverage Test into account, we calculate cumulative probability:

$$Pr = \sum_{k=0}^{91} \binom{2263}{91} 0.95^k 0.05^{2263-k} = 0.016.$$

Such value falls into green zone.

Taking Kupiec's test into account, we calculate the proportion of exceedances $\frac{91}{2263} = 4.02\%$.

$$LR = 2\left(\ln\left((0.0402)^{91}(1-0.0402)^{2263-91}\right) - \ln\left(0.05^{91}(1-0.05)^{2263-91}\right)\right) = 4.88.$$

The critical value at a 5% confidence level is equal to 3.84. Because $4.88 > 3.84$, we reject null hypothesis and consider that model VaR as invalid.

2.3. Credit risk

Credit risk is associated with losses caused by counterparties that are unwilling or unable to fulfill their contractual obligations. It concerns any borrowers, bond issuers, trade agreement counterparties or counterparties in derivatives. The quantity of credit risk is reflected in:

– assessment of the borrower's creditworthiness (rating),
– the amount of the premium (higher interest) that borrower must pay for funds,
– the market price of debt.

Example. Suppose that two companies AAACorp and BBBCorp, are going to issue a 1-year zero-coupon bonds with the notional of \$100 million. Because they have different credit ratings, yield of AAACorp's bond is 4% and yield of BBBCorp's bond is 6%. Today market prices of debt are equal to \$100 million / 1.04 = \$96.15 million for AAACorp and \$100 million / 1.06 = = \$94.34 million for BBBCorp.

Drivers of credit risk

During credit risk measurement, the following decomposition of loss is considered:

$$L = EAD \cdot LGD \cdot b,$$

where:

EAD – Exposure At Default is the economic value of the claim on the counterparty,

LGD – Loss Given Default is the fractional loss (given as a percent of EAD) on obligation which is lost by default, some portion of obligation (recovery rate, *RR*) is recovered by creditors. It holds that $LGD + RR = 1$,

b – is random variable that takes the value 1 if default appears and 0 otherwise.

Random variable *b* has Bernoulli distribution with success probability of *DP* (default probability). Taking expectation and variance (*V*) of *b* we have:

$$E[b] = 1 \cdot DP + 0(1 - DP) = DP,$$

$$V[b] = 1^2 \cdot DP + 0^2(1 - DP) - DP^2 = DP(1 - DP).$$

Assuming that *EAD* is not random variable and that *LGD* and *b* are independent, we have the following expected value and standard deviation (*SD*) of loss:

$$E[L] = EAD \cdot E[LGD] \cdot DP,$$

$$SD[L] = EAD \cdot \sqrt{DP \cdot V[LGD] + DP(1-DP)\left(E[LGD]\right)^2}.$$

Proving the formula for $SD[L]$ requires taking the advantage of the law of total variance – see (Jorion, 2011, p. 457). $E[L]$ is called the expected credit loss but $SD[L]$ represents the unexpected credit loss.

Example. The bank has lent \$1 million to company. The company has 2% chance of defaulting over a year. Some calculations suggest that if it defaults, the bank will recover 60% of the loan. Variance of *LGD* is 50%. Calculate the expected and unexpected credit loss.

$$E[L] = \$1 \text{ million} \cdot (1-0.6) \cdot 0.02 = \$8{,}000,$$

$$SD[L] = \$1 \text{ million} \cdot \sqrt{0.02 \cdot 0.5 + 0.02(1-0.02)(1-0.6)^2} = \$114{,}612.4.$$

2.3.1. Credit ratings

Credit ratings play an important role in the assessment of debt quality. They provide an independent evaluation of the creditworthiness of debt securities issued by governments and corporations. There are three the biggest rating agencies (called the Big Three): Standard and Poor's Agency, Fitch Rating and Moody's Investors Services, they control approximately 95% of the ratings business. Credit agencies take into account many different factors in the assessment process. To assess sovereign bonds they consider, among others, political risk factors, economic structure and economic growth prospects, fiscal flexibility, the public debt level, budget deficit or monetary stability. To assess creditworthiness of corporations, rating agencies analyze financial reports, and economic prospects as well. Among others, they include growth potential, economic environment and exposure to financial risk factors. They also use private information obtained from meetings with management staff. Gradations of creditworthiness are indicated by various rating symbols. Table 4 demonstrates such grades.

The credit quality of most issuers and their obligations is not fixed and steady over time, but tends to undergo changes. A change in rating may occur at any time when an agency observes some alteration in creditworthiness. Thanks to publicly available ratings, investors can easily compare the creditworthiness of many countries. Table 5 shows sovereign credit ratings of 26 countries in March 2018.

Table 4. Rating symbols

Moody's	S&P	Fitch	Rating description
Investment grades: high credibility			
Aaa	AAA	AAA	Prime
Aa1	AA+	AA+	High grade
Aa2	AA	AA	
Aa3	AA–	AA–	
A1	A+	A+	Upper medium grade
A2	A	A	
A3	A–	A–	
Baa1	BBB+	BBB+	Lower medium grade
Baa2	BBB	BBB	
Baa3	BBB–	BBB–	
Speculative grades: low credibility			
Ba1	BB+	BB+	Lower medium grade
Ba2	BB	BB	
Ba3	BB–	BB–	
B1	B+	B+	Non-investment grade speculative
B2	B	B	
B3	B–	B–	
Highly speculative grades: high probability of default			
Caa1	CCC+		Substantial risk
Caa2	CCC	CCC	
Caa3	CCC–		
Ca	CC	CCC	Extremely speculative
	C	CCC	Default imminent
C	C–	DDD	In default
		DD–	
	D	D	

Source: (Wikipedia).

Table 5. Sovereign credit ratings

COUNTRY	S&P RATING	FITCH RATING	MOODY's RATING
AZERBAIJAN	BB+	BBB–	Ba2
BELARUS	B–	NR	Caa1
BRAZIL	BB–	BB	Ba2
CHINA	A+	A+	A1
CROATIA	BB	BB+	Ba2

Table 5 – cont.

COUNTRY	S&P RATING	FITCH RATING	MOODY's RATING
CZECH REPUBLIC	AA–	A+	A1
DENMARK	AAA	AAA	Aaa
EGYPT	B–	B	B3
FRANCE	AA	AA	Aa2
GERMANY	AAA	AAA	Aaa
GREECE	B	B	B3
HUNGARY	BBB–	BBB–	Baa3
INDIA	BBB–	BBB–	Baa2
IRELAND	A+	A+	A2
ITALY	BBB	BBB	Baa2
JAPAN	A+	A	A1
NIGERIA	B	BB–	B1
NORWAY	AAA	AAA	Aaa
POLAND	BBB+	A–	A2
PORTUGAL	BB	BBB	Ba1
RUSSIA	BBB–	BBB–	Ba1
SLOVAKIA	A+	A+	A2
SPAIN	BBB+	A–	Baa2
UKRAINE	B–	B–	Caa2
U.S.	AA+	AAA	Aaa
VIETNAM	BB–	B+	B1

Note: NR – no rating.

Source: (Wikipedia).

2.3.2. Internal credit ratings

In practice only large bond issuers receive ratings from credit rating agencies but not companies that are not publicly traded. Most banks have their own procedures to describe creditworthiness of their counterparties like small and medium-sized firms. Such internal ratings are based on estimating the risk of defaulting on the basis of financial statements indices. The most widespread tool is Altman's *Z*-score developed in 1968. It is a statistical technique known as discriminant analysis aiming at bankruptcy prediction. The *Z*-score formula is:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5,$$

where:

X_1 – working capital/total assets,
X_2 – retained earnings/total assets,
X_3 – earnings before interest and taxes/total assets,
X_4 – market value of equity/book value of total liabilities,
X_5 – sales/total assets.

There are three zones of discrimination. The "safe" zone is if $Z > 2.99$ and the company is considered unlikely to default. The "grey" zone is if $1.81 < Z < 2.99$ and it is a good chance of the company going bankrupt within the next 2 years of operations. The "distress" zone is for $Z < 1.81$, then the score indicates a high probability of distress within this time period. In its initial test, the Altman's Z-score was found to be 72% accurate in predicting bankruptcy two years prior to the event. The above Z-score formula is dedicated to publicly traded manufacturing companies. Later variations by Altman were designed to be applicable to privately held companies and non-manufacturing companies A lot of similar models were developed over years to reflect specific factors of firms, markets or countries.

Example. Consider a company for which current assets are 150,000, current liabilities are 50,000, total assets are 350,000, retained earnings is 110,000, earnings before interest and taxes is 35,000, sales are 250,000, market value of equity is 220,000 and book value of total liabilities is 120,000.

Calculating Z-score factors we have

$$X_1 = \frac{150{,}000 - 50{,}000}{350{,}000} = 0.286,\ X_2 = \frac{110{,}000}{350{,}000} =$$

$$= 0.314, X_3 = \frac{35{,}000}{350{,}000} = 0.1, X_4 = \frac{220{,}000}{120{,}000} = 1.833, X_5 = \frac{250{,}000}{350{,}000} = 0.714,$$

$$Z = 1.2 \cdot 0.286 + 1.4 \cdot 0.314 + 3.3 \cdot 0.1 + 0.6 \cdot 1.833 + 0.999 \cdot 0.714 = 2.926.$$

Because $Z < 2.99$ the company is in danger of bankruptcy in the next 2 years.

2.3.3. Measuring default risk probabilities from bond spreads

Credit spread is a difference between bond yield and risk-free rate: $y - y_f$. Credit spread is a risk premium and a valuable source of information about credit risk of obligation. Suppose that an investor has a one-year zero-coupon bond. Next year the bond will pay off the notional, N if the bond is not defaulting, or otherwise only the recovery rate (RR) of notional. In "risk-neutral world" the current price of asset is equal to expected value of payoff discounted with risk-free interest rate

$$\frac{N}{1+y} = \frac{N}{1+y_f}(1-DP) + \frac{N \cdot RR}{1+y_f}DP.$$

After rearranging terms we have:

$$\left(1+y_f\right) = (1+y)(1-DP \cdot LGD),$$

thus implied default probability equals:

$$DP = \frac{1}{LGD}\left(1 - \frac{1+y_f}{1+y}\right) = \frac{1}{LGD}\left(\frac{y-y_f}{1+y}\right).$$

Rewriting the upper equation, we can show credit spread as follows:

$$y - y_f = LGD \cdot DP(1+y).$$

We can conclude that credit spread approximately is a product of yield and loss given default. The above equations explain how credit spread determines the probability of default, higher spread entails greater probability of default in future.

Similarly one can show that for multiple, T-period, case-implied default probability equals:

$$DP = \frac{1}{LGD}\left(1 - \left(\frac{1+y_f}{1+y}\right)^T\right),$$

where DP is the average annual default probability.

Example. Consider a zero-coupon bond with the notional of $1,000 and maturity in 5 years. Bond yield is 6% and risk-free rate is 2%. Calculate the default probability supposing that recovery rate is 40%.

$$DP = \frac{1}{(1-0.4)}\left(1-\left(\frac{1+0.02}{1+0.06}\right)^5\right) = 29.16\%.$$

An average probability of defaulting in 5 years equals 29.16%.

2.3.4. Measuring default risk probabilities from equity prices

In 1974, Robert Merton proposed a method to evaluate credit risk of a company by modeling the company's equity as a call option on its assets. Under the model, a default event occurs when firm's assets reach a sufficiently low level compared to its liabilities. The model is called structural because it provides a relationship between the credit risk and the financial structure of the firm. The market value of a company's assets V_t at time t is a sum of company's equity E_t, and market value B_t of its liabilities L with maturity T:

$$V_t = E_t + B_t.$$

If at maturity date T, $V_T \geq L$ the value of shareholders equals $V_T - L$. However, if $V_T < L$ shareholders receive nothing and the firms defaults because in this case additional funds are necessary to pay company's debt. Equity of the company is then as follows:

$$E_T = \max\left(V_T - L, 0\right).$$

The above equation shows that company's equity at maturity is a call option with the value of assets as underlying and liability level as a strike price. In practice, firms have many liabilities with different maturities, and liability threshold is commonly set somewhere between the value of the short-term liabilities and the value of the total liabilities e.g. short-term liabilities plus a half of long-term liabilities.

Rearranging the above equations, we get the market value of debt:

$$B_T = V_T - E_T = V_T - \max\left(V_T - L, 0\right) = L - \max\left(L - V_T, 0\right).$$

It is equivalent to a long position in a risk-free zero-coupon bond and a short position in a put option with the value of assets as underlying and liability level as a strike price. We can illustrate equity and market value of debt in Figure 4.

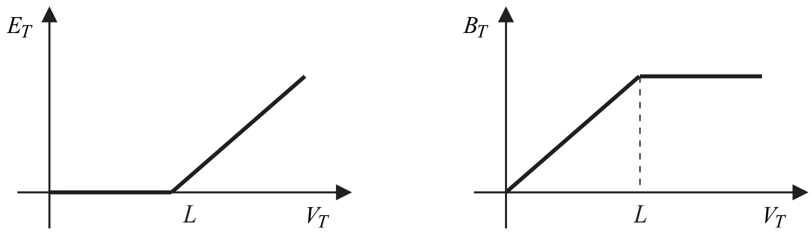

Figure 4. Values of equity and debt

Taking the above information and Black-Scholes pricing formula we obtain:

$$E_t = V_t N(d_1) - Le^{-r(T-t)} N(d_2),$$

$$B_t = V_t - E_t = V_t N(-d_1) + Le^{-r(T-t)} N(d_2),$$

where:

$$d_1 = \frac{\ln\left(\frac{V_t}{L}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$N(\cdot)$ – cumulative standard normal distribution function,
r – risk-free interest rate,
σ – volatility of firm's assets $(\sigma = \sigma_V)$.

The market value of the firm's assets is an unobservable variable–it is not possible to calculate directly its volatility. Only the market value of equity is observable, given by the firm's stock market price times the number of outstanding shares. The Merton model combines volatility of firm's assets and volatility of equity with the equation:

$$dE_t = \frac{\partial E_t}{\partial V_t} dV_t.$$

Derivative $\frac{\partial E_t}{\partial V_t}$ is an option delta $\left(N(d_1)\right)$ in Black-Scholes model, thus:

$$dE_t = N(d_1)dV_t \Rightarrow E_t \frac{dE_t}{E_t} = V_t N\left(d_1\right)\frac{dV_t}{V_t}.$$

Taking the volatility, we obtain a relationship that allows us to calculate the volatility of firm's assets:

$$E_t \sigma_E = V_t N\left(d_1\right)\sigma_V.$$

Wanting to solve problem one has to solve the system of equations:

$$\begin{cases} E_t = V_t N\left(d_1\right) - Le^{-r(T-t)} N\left(d_2\right), \\ E_t \sigma_E = V_t N\left(d_1\right)\sigma_V, \end{cases}$$

with respect to V_t and σ_V.

From Black-Scholes model we can estimate risk-neutral default probability that equals:

$$DP = Pr\left(V_T < L\right) = N\left(-d_2\right).$$

Credit losses at maturity T are equal to $L - B_T$. Expected credit losses are priced as the difference of liability and the future value of market value of debt:

$$E\left[L_t\right] = L - B_t e^{r(T-t)} = L - V_t N\left(-d_1\right)e^{r(T-t)} - L\,N\left(d_2\right) = -V_t N\left(-d_1\right)e^{r(T-t)} +$$

$$+N\left(-d_2\right)L = L\left(1 - \frac{V_t}{L}\frac{N\left(-d_1\right)}{N\left(-d_2\right)}e^{r(T-t)}\right)N\left(-d_2\right).$$

Because $EAD = L$ and $DP = N\left(-d_2\right)$ and, as it was indicated at the beginning of this subchapter, $E[L] = EAD \cdot E[LGD] \cdot DP$, we can compare the middle terms:

$$E[LGD] = 1 - \frac{V_t}{L}\frac{N\left(-d_1\right)}{N\left(-d_2\right)}e^{r(T-t)}.$$

We can obtain a recovery rate as an endogenous variable in the model:

$$RR = \frac{V_t}{L} \frac{N(-d_1)}{N(-d_2)} e^{r(T-t)}.$$

Example. Consider a company that has equity of $5 million and volatility of the equity of 60%. The liability that will have to be paid in two-year horizon is $10. The risk free rate is 5% per annum.

In this case we have $E_t = 5, \sigma_E = 0,6, r = 5\%, T = 2$. Solving the system of equations we obtain $V_t = 13.88$ and $\sigma_V = 0.233$. Probability of default is 12.8%. The market value of the debt is 8.88. Expected loss on the debt is therefore 18.37% and recovery rate is 85.65%.

Further readings

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CHAPTER 3
INTRODUCTION TO DERIVATIVE INSTRUMENTS PRICING

3.1. Pricing in one-period model

3.1.1. Stochastic model of market

We will consider finite models of markets – i.e. models with discrete time in which prices of all the available assets can take values from a finite set of numbers. In this subchapter we will study the simplest version of such a model, namely we consider only two trading dates. This is obviously very unrealistic oversimplification of real changes in stock and bond prices, but it allows to point out some important features of stochastic models of markets and develop basic relationships that hold true also in much more complicated models.

We assume that there are only two trading dates: the initial date $t=0$ and the terminal date $t=T$. We have all the information about events and prices at the initial date – it is "the present moment". However, we do not know what will happen in the future. The prices at the terminal date are modelled as random variables. To simplify, we assume that the sample space is finite. There are M possible outcomes (or **states of the world**) in the future. The sample space is thus defined as follows:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_M\}.$$

Each state of the world could happen with some positive probability. The probability measure (called **real probability**) of the state of the world ω is given by $P(\omega) > 0$. Formally the stochastic setup of the model is defined by finite probability space (Ω, F, P), where F is a σ-algebra of all subsets of Ω.

In the financial market there are $N+1$ financial assets, which are labelled form 0 to N. The prices of the asset n at the moment t is given by $S_n(t)$, where $n=0,1,\ldots,N$, $t=0,T$. We assume that all prices are non-negative (at the moment $t=0$ and at the moment $t=T$ in all possible states of the world). The prices at the moment $t=T$ can be different in different states of the world.

The prices of the first instrument, $S_0(t)$, are strictly positive. We take this instrument as a **numéraire** – the values of all the other instruments and portfolios will be measured in the units of the numéraire. Traditionally, a numéraire is assumed to be a riskless security and one uses the terms “bond” or “bank account” to describe it. In this approach, if the risk-free interest rate from the moment 0 to the final moment T equals $r > 0$, then the prices of the first instrument are given by

$$S_0(0)=1 \quad \text{and} \quad S_0(T)=1+r.$$

Generally, a numéraire does not have to be riskless. However, without losing generality we can assume that $S_0(0)=1$. Let us define the discount factor as: $\beta(t)=1/S_0(t)$. Thus $\beta(0)=1$ and $\beta(T)=\frac{1}{1+r}$.

The prices at the initial moment are known to the investor, but the prices at the moment T are random variables. Using vector notation we can write:

$$S(t)=\left(S_0(t), S_1(t), \ldots, S_N(t)\right)^T.$$

$S(0)$ is a $(N+1)$-dimensional vector and $S(T)$ is $(N+1)$-dimensional random variable. The prices of assets at the terminal date are random variables and depend on the state of the world. We state this dependence explicitly expressing the prices in the state of the world ω by the vector $S(T,\omega)=\left(S_0(t,\omega),S_1(t,\omega), \ldots, S_N(t,\omega)\right)^T$.

Define also $\tilde{S}(t)=\left(\tilde{S}_0(t), \tilde{S}_1(t), \ldots, \tilde{S}_N(t)\right)^T=\left(1, \beta(t)S_1(t), \ldots, \beta(t)S_N(t)\right)^T$ as the vector of **discounted** assets’ prices, i.e. prices measured in the units of numéraire. Again, $\tilde{S}(0)$ is $(N+1)$-dimensional vector and $\tilde{S}(T)$ is a vector random variable and its value depends on ω: $\tilde{S}(T)=\tilde{S}(T,\omega)=\left(1, \tilde{S}_1(T,\omega), \ldots, \tilde{S}_N(T,\omega)\right)^T$.

A **portfolio** or a **trading strategy** h is an $(N+1)$-dimensional vector that describes the holdings of the investor, $h=\left(h_0, h_1, \ldots, h_N\right)$. Here h_n denotes the number of asset n held in the portfolio from the moment $t=0$ till terminal date T. The **value** of the portfolio at the moment t is given by

$$V^h(t)=h\cdot S(t)=\sum_{n=0}^{N} h_n S_n(t).$$

The value at the initial moment is a constant, whereas the value at the final moment is a random variable: $V^h(T,\omega)=h\cdot S(T,\omega)=\sum_{n=0}^{N} h_n S_n(T,\omega)$.

The **gain** from the portfolio h equals

$$G^h = V^h(T)-V^h(0)=h\cdot \Delta S=\sum_{n=0}^{N} h_n \Delta S_n,$$

where $\Delta S_n = S_n(T)-S_n(0)$ and $\Delta S=\left(\Delta S_0,\ldots,\Delta S_N\right)^T$. We define also the **discounted value** of the portfolio – i.e. the value of the portfolio measured in the units of the numéraire:

$$\tilde{V}^h(t)=\beta(t)V^h(t)=h\cdot\tilde{S}(t)=\sum_{n=0}^{N} h_n \tilde{S}_n(t).$$

The **discounted gain** is defined as

$$\tilde{G}^h=\tilde{V}^h(T)-\tilde{V}^h(0)=h\cdot\Delta\tilde{S}=\sum_{n=1}^{N} h_n \Delta\tilde{S}_n,$$

where $\Delta\tilde{S}_n=\tilde{S}_n(T)-\tilde{S}_n(0)$. Notice that summation starts with $n=1$ as the increment in the discounted value of the numéraire is 0.

Example 1

Assume that there are three assets. The numéraire is the bank account and the riskless interest rate equals 5%. Thus $S_0(0)=1$ and $S_0(T)=1.05$. The discount factor equals $\frac{1}{1.05}=0.95238$. The prices of risky assets at the initial moment are $S_1(0)=50$ and $S_2(0)=150$. We assume that there are three states of the world, so $\Omega=\{\omega_1,\omega_2,\omega_3\}$. We assume the following probabilities for the states of the world: $P(\omega_1)=\frac{1}{2}$ and $P(\omega_2)=P(\omega_3)=\frac{1}{4}$. The final prices of risky assets in different states of the world are given in Table 1.

The discounted prices of risky assets in all the possible states of the world are given in Table 2.

Table 1. Data for example 1–prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$S_0(T,\omega)$	1.05	1.05	1.05
$S_1(T,\omega)$	47.25	47.25	63
$S_2(T,\omega)$	162.75	157.5	152.25

Table 2. Example 1–discounted prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$\tilde{S}_0(T,\omega)$	1	1	1
$\tilde{S}_1(T,\omega)$	45	45	60
$\tilde{S}_2(T,\omega)$	155	150	145

The increments in the prices and discounted prices are given in Table 3.

Table 3. Example 1–increments in the prices and discounted prices

	ϖ_1	ϖ_2	ϖ_3
$\Delta S_0(\omega)$	0.05	0.05	0.05
$\Delta S_1(\omega)$	–2.75	–2.75	13
$\Delta S_2(\omega)$	12.75	7.5	2.25

	ϖ_1	ϖ_2	ϖ_3
$\Delta\tilde{S}_0(\omega)$	0	0	0
$\Delta\tilde{S}_1(\omega)$	–5	–5	10
$\Delta\tilde{S}_2(\omega)$	5	0	–5

A portfolio is a three-dimensional vector $h=(h_0,h_1,h_2)^T$. Its value at the initial moment equals $V^h(0)=h_0+50h_1+150h_2$. The value at the terminal moment depends on the state of the world. In Table 4 the values, discounted values, gains and discounted gains are calculated for every possible state of the world.

Table 4. Example 1–values and gains

	ϖ_1	ϖ_2	ϖ_3
$V^h(T,\omega)$	$1.05h_0+47.25h_1+162.75h_2$	$1.05h_0+47.25h_1+157.5h_2$	$1.05h_0+63h_1+152.25h_2$
$\tilde{V}^h(T,\omega)$	$h_0+45h_1+155h_2$	$h_0+45h_1+150h_2$	$h_0+60h_1+145h_2$
$G^h(\omega)$	$0.05h_0-2.75h_1+12.75h_2$	$0.05h_0-2.75h_1+7.5h_2$	$0.05h_0+13h_1+2.25h_2$
$\tilde{G}^h(\omega)$	$-5h_1+5h_2$	$-5h_1$	$10h_1-5h_2$

Example 2

We assume, as in Example 1, that interest rate is 5% and the prices of two risky assets at the initial date are 50 and 150. There are three equally possible states of the world and the prices at the terminal date are given in Table 5.

Table 5. Data for example 2–prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$S_1(T,\omega)$	42	47.25	63
$S_2(T,\omega)$	162.75	157.5	152.25

Try to calculate discounted prices, increments and discounted increments as well as values and gains (and their discounted counterparts). The answers are given in Table 6 and Table 7.

Table 6. Discounted prices and increments in Example 2

	ϖ_1	ϖ_2	ϖ_3
$\tilde{S}_0(T,\omega)$	1	1	1
$\tilde{S}_1(T,\omega)$	40	45	60
$\tilde{S}_2(T,\omega)$	155	150	145
$\Delta S_0(\omega)$	0.05	0.05	0.05
$\Delta S_1(\omega)$	–8	–2.75	13
$\Delta S_2(\omega)$	12.75	7.5	2.25

	ϖ_1	ϖ_2	ϖ_3
$\Delta\tilde{S}_0(\omega)$	0	0	0
$\Delta\tilde{S}_1(\omega)$	–10	–5	10
$\Delta\tilde{S}_2(\omega)$	5	0	–5

Table 7. Values and gains in Example 2

	ϖ_1	ϖ_2	ϖ_3
$V^h(T,\omega)$	$1.05h_0+42h_1+162.75h_2$	$1.05h_0+47.25h_1+157.5h_2$	$1.05h_0+63h_1+152.25h_2$
$\tilde{V}^h(T,\omega)$	$h_0+40h_1+155h_2$	$h_0+45h_1+150h_2$	$h_0+60h_1+145h_2$
$G^h(\omega)$	$0.05h_0-8h_1+12.75h_2$	$0.05h_0-2.25h_1+7.5h_2$	$0.05h_0+13h_1+2.25h_2$
$\tilde{G}^h(\omega)$	$-10h_1+5h_2$	$-5h_1$	$10h_1-5h_2$

Example 3

Let us take all the data from Example 1 and assume that there is one additional state of the world ω_4. The sample space is now $\Omega=\{\omega_1,\ldots,\omega_4\}$. We assume that all outcomes are equally probable, thus $P(\omega_i)=\frac{1}{4}$. The prices of risky assets at the terminal date in the new state of the world are $S_1(T,\omega_4)=68.25$ and $S_2(T,\omega_4)=147$. The discounted prices in the additional state of the world are $\tilde{S}_1(T,\omega_4)=65$ and $\tilde{S}_2(T,\omega_4)=140$. The increments are $\Delta S_1(\omega_4)=18.25$, $\Delta S_2(\omega_4)=-3$, $\Delta\tilde{S}_1(\omega_4)=15$ and $\Delta\tilde{S}_2(\omega_4)=-10$. The value and discounted value are $V^h(T,\omega_4)=1.05h_0+68.25h_1+147h_2$ and $\tilde{V}^h(T,\omega_4)=h_0+65h_1+140h_2$.

The gain and discounted gain equal $G^h(\omega_4)=0.05h_0+18.25h_1-3h_2$ and $\tilde{G}^h(\omega_4)=15h_1-10h_2$.

3.1.2. Arbitrage opportunity and the First Fundamental Theorem of Asset Pricing

We assume that the market is "frictionless". There are no transaction costs or taxes, an investor can build any portfolio he wishes – there are no restrictions on the size of position, unlimited short sales and borrowing are allowed. Additionally, the securities are perfectly divisible, which means that the investor's positions h_i can take any real values.

To be economically reasonable, the model should fulfill some additional assumptions. In particular, the model is unreasonable if it assumes that the investor is able to make profits without any exposure to risk. Such a possibility would be "a free lunch" and it is assumed that it is impossible in the real market. Formally, we define this as arbitrage opportunity.

A portfolio h is an **arbitrage opportunity** (or **arbitrage strategy**) if its initial value is zero, $V^h(0)=0$, and its value at the terminal date satisfies

$$P\left(V^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(V^h(T)>0\right)>0.$$

The arbitrage opportunity is thus a portfolio with an initial value, which almost surely (i.e. with probability 1) produces a non-negative final value and, with positive probability, its final value can be positive. In this definition the nominal (not discounted) values were used. However, one can also use a discounted value or discounted gain to define the arbitrage opportunity.

In the model there exists an arbitrage opportunity if and only if there exists a portfolio h such that $\tilde{V}^h(0)=0$, and at the terminal date its discounted value fulfills:

$$P\left(\tilde{V}^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{V}^h(T)>0\right)>0$$

or there is a portfolio h such that

$$P\left(\tilde{G}^h\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{G}^h>0\right)>0.$$

The last criterion, which uses discounted gain, is the easiest one to check. Note that in this criterion there is no assumption that initial value of the arbitrage strategy should equal zero. If there is no arbitrage opportunity in the model, we say that the model is **arbitrage-free** or **viable**.

Example 1 – cont.

We will consider if there exists an arbitrage opportunity in the model presented in Example 1. To this end we will make use of the last criterion based on discounted gain. Discounted gains in all states of the world are calculated in Table 3 (the last row). As all states of the world have positive probability, for the arbitrage strategy we should have $\tilde{G}^h(\omega) \geq 0$ for every ω. An arbitrage opportunity exists if and only if the following system of inequalities

$$-5h_1 + 5h_2 \geq 0,$$

$$-5h_1 \geq 0,$$

$$10h_1 - 5h_2 \geq 0$$

has a solution with at least one inequality being strict. As it is easy to check, the only solution to this system is $h_1 = h_2 = 0$. Thus in this model there is no arbitrage opportunity.

Example 2 – cont.

One can easily check that for any portfolio with $h_1 = -1$ and $h_2 = -2$ we have $\tilde{G}^h(\omega_1) = \tilde{G}^h(\omega_3) = 0$ and $\tilde{G}^h(\omega_2) = 5 > 0$. Thus the arbitrage opportunity exists in this model. If we take $h_0 = 350$, we obtain the portfolio $h = (350, -1, -2)$ such that $V^h(0) = 0$, $V^h(T, \omega_1) = V^h(T, \omega_3) = 0$ and $V^h(T, \omega_2) = 5.25$. This portfolio is thus an arbitrage strategy.

Let us consider the same model of market as before, but assume that instead of real probabilities $P(\omega)$ of different states of the world, we have some artificial probabilities $Q(\omega)$. We assume that the two probabilistic measures, P and Q, are equivalent, which means that $Q(\omega) > 0$ for all the states of the world with $P(\omega) > 0$. In our setup this is equivalent to the assumption that $Q(\omega) > 0$ for all $\omega \in \Omega = \{\omega_1, \ldots, \omega_M\}$.

A probabilistic measure Q is called a **martingale measure** (or **risk neutral measure**) if the initial prices of all instruments are equal to the expected values (calculated with respect to Q) of their discounted terminal prices:

$$S_n(0) = E^Q\left[\tilde{S}_n(T)\right] = E^Q\left[\beta(T) S_n(T)\right] = \sum_{i=1}^{M} Q(\omega_i)\, \beta(T) S_n(T, \omega_i). \quad (1)$$

In the formula above $E^Q[\cdot]$ is an expected value calculated for the probability measure Q. It should be noted that the martingale measure is always connected with the numéraire that is used. We can always change the numéraire and, if there exists a martingale measure for standard numéraire S_0, there is also a probability measure in which initial prices of all securities (measured in the units of numéraire) are expected values of their prices. For example, suppose that asset 1 can serve as a numéraire (i.e. its final prices is always positive, $S_1(T) > 0$). Let us define probability measure Q^1 as follows:

$$Q^1(\omega) = \frac{S_1(T,\omega)}{S_1(0)S_0(T,\omega)} Q(\omega). \tag{2}$$

One can easily check that probabilities Q^1 are positive and that they sum up to 1:

$$\sum_{i=1}^{M} Q^1(\omega_i) = \sum_{i=1}^{M} \frac{S_1(T,\omega_i)}{S_1(0)S_0(T,\omega_i)} Q(\omega_i) = \frac{1}{S_1(0)} \sum_{i=1}^{M} \tilde{S}_1(T,\omega_i) Q(\omega_i) = \frac{E^Q\left[\tilde{S}_1(T,\omega)\right]}{S_1(0)} =$$
$$= \frac{S_1(0)}{S_1(0)} = 1.$$

If we denote by $E^1[\cdot]$ the expected value calculated for the probability measure Q^1, we can make the following derivations:

$$S_1(0)E^1\left[\frac{S_n(T)}{S_1(T)}\right] = S_1(0)\sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_1(T,\omega_i)} Q^1(\omega_i) =$$
$$= S_1(0)\sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_1(T,\omega_i)} \frac{S_1(T,\omega_i)}{S_1(0)S_0(T,\omega_i)} Q(\omega_i) =$$
$$= \sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_0(T,\omega_i)} Q(\omega_i) = E^Q\left[\tilde{S}_n(T)\right] = S_n(0).$$

Thus we have:

$$\frac{S_n(0)}{S_1(0)} = E^1\left[\frac{S_n(T)}{S_1(T)}\right],$$

which is an equivalent of equation (1) for the asset 1 as a numéraire. The probabilities Q^1 are called martingale measure for the numéraire S_1. Similarly, one can define martingale measures Q^2, Q^3, … and so on–for any security that can serve as a numéraire. The martingale measure Q, connected with the standard numéraire S_0 will be called simply "risk-neutral measure" or "martingale measure" (without any addition).

There exists a deep relationship between the existence of arbitrage opportunity in the model and martingale measures. It is known as "First Fundamental Theorem", which states that existence of an arbitrage opportunity and existence of martingale measure are mutually exclusive.

First Fundamental Theorem. The model of financial market is arbitrage-free if and only if there exists a martingale measure, i.e. the probability measure Q for which the following equation holds for all securities:

$$S_n(0) = E^Q\left[\frac{S_n(T)}{S_0(T)}\right] = E^Q\left[\beta(T) S_n(T)\right]. \tag{3}$$

This rule can be also reformulated as follows. The model is arbitrage-free if for any security S_m that can serve as a numéraire there exists a probability measure Q^m such that

$$S_n(0) = S_m(0) E^m\left[\frac{S_n(T)}{S_m(T)}\right]. \tag{4}$$

The economic interpretation of equations (3) and (4) is that initial prices of securities are obtained as the expected value (under appropriate probability measure Q or Q^m) of the final prices of assets, discounted with the chosen numéraire. The equation (3) is also referred to as the "risk-neutral" pricing formula. The proof of the theorem is rather simple and is based on the mathematical result known as "separating hyperplane theorem". The proof can be found, e.g. in (Bingham & Kiesel, 2004; Elliott & Kopp, 1999; Pliska, 1997).

The other question is how to find martingale probabilities, if such a measure exists in the model. One can try to calculate them directly from the definition – using equations (1) or (3). We can also transform these equations. Using the definition of increments of discounted prices, $\Delta\tilde{S}_n = \tilde{S}_n(T) - \tilde{S}_n(0)$, we obtain the following condition:

$$E^Q\left[\Delta\tilde{S}_n\right] = 0 \quad \text{for all } n = 1, \ldots, N. \tag{5}$$

Example 1 – cont.

We will try to obtain a risk-neutral measure in the model presented in Example 1. To this end, we will use the condition (4). Let us denote by q_1, q_2 and q_3 martingale probabilities of the states of the world ω_1, ω_2 and ω_3, respectively. Rewriting equation (5) for $n=1, 2$ and using the condition that probabilities should sum up to 1, we obtain the following system of linear equations:

$$-5q_1 - 5q_2 + 10q_3 = 0,$$

$$5q_1 - 5q_3 = 0,$$

$$q_1 + q_2 + q_3 = 1.$$

The only solution to this system is $q_1 = q_2 = q_3 = \frac{1}{3}$. Thus in the martingale measure all states of the world are equally probable. One can easily check that the martingale probabilities fulfill also equation (3). In the model there is a martingale measure, so the model is arbitrage-free, as we have already checked in a more direct way.

Let us consider now taking S_1 as a numéraire. Using equation (2) we can calculate the martingale measure for this numéraire. Denote by q_1^1, q_2^1 and q_3^1 the martingale probabilities of subsequent states of the world. We have:

$$q_1^1 = q_2^1 = \frac{47.25}{50 \cdot 1.05} \cdot \frac{1}{3} = 0.3,$$

$$q_3^1 = \frac{63}{50 \cdot 1.05} \cdot \frac{1}{3} = 0.4.$$

One can easily check that the obtained probabilities indeed are a martingale measure for the numéraire S_1, i.e. they fulfill condition (3). For example, taking $n=2$ we obtain:

$$S_1(0)E^1\left[\frac{S_2(T)}{S_1(T)}\right] = 50\left(\frac{162.75}{47.25} \cdot 0.3 + \frac{157.5}{47.25} \cdot 0.3 + \frac{152.25}{63} \cdot 0.4\right) = 50 \cdot 3 = 150 = S_2(0).$$

To check this condition for security S_0 is left as an exercise.

Example 2 – cont.

One can use the equation (5) to form a system of linear equations, similar to that in example 1. The only solution to the system is $q_1=\frac{1}{2}$, $q_2=0$, $q_3=\frac{1}{2}$. The only measure that satisfies the condition (5) is thus $Q(\omega_1)=Q(\omega_3)=\frac{1}{2}$, $Q(\omega_2)=0$. This is a well-defined probability measure, but it fails to satisfy the assumption that a martingale measure should be equivalent to the real measure P. In this case measure Q is not a risk-neutral measure because one of the probabilities is not positive. Thus, according to the First Fundamental Theorem, the model is not arbitrage-free.

Example 3 – cont.

In the model presented in Example 3 the discounted increments of prices in the states of the world $\omega_1, \ldots, \omega_3$ are the same as in Example 1. The discounted increments of prices in state of the world ω_4 are $\Delta\tilde{S}_1(\omega_4)=15$ and $\Delta\tilde{S}_2(\omega_4)=-10$. The condition (5) leads to the following system of equations:

$$-5q_1-5q_2+10q_3+15q_4=0,$$

$$5q_1-5q_3-10q_4=0,$$

$$q_1+q_2+q_3+q_4=1\,.$$

The system has more than one solution. Solving it, one can obtain that set of the solutions:

$$\left\{\left(q,\frac{1}{2}-\frac{1}{2}q,1-2q,\frac{3}{2}q-\frac{1}{2}\right):q\in\left(\frac{1}{3},\frac{1}{2}\right)\right\}.$$

In this case there is no single martingale measure. There are many such measures – in fact, infinitely many. All martingale measures can be expressed as a combination of two “extreme” measures Q_0 and Q_1, where $Q_0(\omega_1)=Q_0(\omega_2)=Q_0(\omega_3)=\frac{1}{3}$, $Q_0(\omega_4)=0$ and $Q_1(\omega_1)=\frac{1}{2}$, $Q_1(\omega_3)=0$, $Q_1(\omega_2)=Q_1(\omega_4)=\frac{1}{4}$. Then every measure Q_λ, defined as $Q_\lambda(\omega)=(1-\lambda)Q_0(\omega)+\lambda Q_1(\omega)$ for $\lambda\in(0,1)$, is a martingale measure. Note that the measures Q_0 and Q_1 are not martingale measures, as they assign probability zero to some states of the world.

3.1.3. Replication and completeness of the market

In the model we have $N+1$ securities, whose initial prices are given and whose values at the terminal date can be random. Usually one assumes than the terminal price of the first instrument (S_0) is known in advance and the future prices of all other N instruments are random. We assume that these securities are correctly priced by the market. We will consider the following problem. We consider a financial instrument that was not traded in the market previously. We assume that cash flows connected with this new instrument will take place at the terminal moment T and that these cash flows can depend on the state of the world. The seller of the instrument promises the buyer to pay him at the date T the amount $X(\omega)$ if the true state of the world turns out to be ω. If $X(\omega)$ is negative, we interpret it as the sum that the buyer is to pay the seller. Thus in the further considerations we can treat a **derivative instrument** or **contingent claim** as a random variable defined on Ω:

$$X:\Omega \to R.$$

Example 4

Consider a European call option, which gives the owner a right (but not an obligation) to buy a unit of underlying asset in the future (at execution date T) at a predetermined strike price K. The option will be executed only if at the execution date the price of the underlying asset is higher than the strike price. Otherwise it is worthless to the holder. The payoffs of the option are thus given by the following formula:

$$X(\omega)=\left(S(T,\omega)-K\right)^{+},$$

where $S(T,\omega)$ is the price of the underlying asset at the moment T in the state of the world ω and the plus in the superscript means a nonnegative value of the expression, i.e. for any expression x the symbol x^{+} denotes $x^{+}=\max\{0,x\}$.

On the other hand, a European put option gives the owner a right to sell a unit of underlying asset at a predetermined strike price. The transaction may take place at the specified date (the execution date T). The owner will execute the option only if at the execution date the price of the underlying asset is lower than the price in the contract. The payoff is thus given by

$$X(\omega)=\left(K-S(T,\omega)\right)^{+}.$$

Let us take the model presented in Example 1 and consider three derivative instruments: a call option on the first asset with the strike price $K=50$, a put option with the same strike prices, and an option which gives the right to buy at the terminal date a basket consisting of one unit of asset 1 and one unit of asset 2 for the price 200. We will denote these contingent claims by X^1, X^2 and X^3, respectively. As it is easy to calculate, the payoffs of these claims are

$$X^1(\omega)=\begin{cases} 0, & \omega=\omega_1, \\ 0, & \omega=\omega_2, \\ 13, & \omega=\omega_3, \end{cases} \quad X^2(\omega)=\begin{cases} 2.75, & \omega=\omega_1, \\ 2.75, & \omega=\omega_2, \\ 0, & \omega=\omega_3, \end{cases}$$

$$X^3(\omega)=\begin{cases} 10, & \omega=\omega_1, \\ 4.75, & \omega=\omega_2, \\ 15.25, & \omega=\omega_3. \end{cases}$$

We will also consider the same instruments in the model presented in Example 3, with additional state of the world, ω_4. The payoffs of contingent claim we consider in the additional state of the world are $X^1(\omega_4)=18.25$, $X^2(\omega_4)=0$ and $X^3(\omega_4)=15.25$.

The natural question to ask is the problem of derivative instruments pricing. We know the payoffs that the instrument could bring in the future in different states of the world and we can ask what should be the current fair price of this instrument. The fundamental finding in the pricing of derivative instruments is the notion of replication, which gives the methods for establishing a unique fair price for some of the contingent claims.

A portfolio h **replicates** a contingent claim X if its final value is equal to the payoff of X in all possible states of the world, i.e.

$$V^h(T,\omega)=X(\omega) \text{ for all } \omega\in\Omega. \tag{6}$$

A contingent claim for which there exists a replicating portfolio is called **attainable**. In case of an attainable claim, one can calculate the correct price assuming that the new financial instrument should not create arbitrage opportunities.

Example 5

A forward contract is an agreement which gives its owner the right and obligation to buy an underlying asset in the future (at the moment *T*) for a specified price K. If the underlying asset in the contract is the first security, S_1, then the payoffs of the forward contract are equal to

$$X = S_1(T) - K.$$

Let us assume that the numéraire S_0 is the risk-free bond, whose price at the initial moment is $S_0(0)=1$ and the final price is equal to $S_0(T)=1+r$, where r it the risk-free rate. Consider the following portfolio $h=\left(-\frac{K}{1+r},1\right)$ in which we buy one unit of the underlying asset and borrow $-K/(1+r)$ units of currency. The value of this portfolio at the terminal date T equals $V^h(T)=S_1(T)-K$, so it is equal to the payoff of the forward contract – regardless of the final price of the security S_1.

We claim that the initial price of the forward should be equal to the initial value of the portfolio h, $V^h(0)=S_1(0)-\frac{K}{1+r}$. Indeed, if the initial price of the forward is higher, say $X_0 > V^h(0)$, then one can sell the forward (take a short position in the contract) and use the money to buy the portfolio h. At the terminal date T the money from the portfolio allows to pay for the claims connected with the short position in the contract. The difference $X_0 - V^h(0)$ is a riskless gain for the investor. On the other hand, if the initial price of the forward contract is lower than the initial value of the portfolio h, $X_0 < V^h(0)$, the investor can take a long position in the contract and sell the portfolio h, i.e. create the portfolio with the opposite positions: $-h=\left(\frac{K}{1+r},-1\right)$. In practical terms this means selling short one unit of the security S_1 and investing the amount $\frac{K}{1+r}$ in the riskless asset. At the terminal data the payoffs from the contract and the value of the portfolio cancel each other out. The difference $V^h(0)-X_0$ is the risk-free gain. Thus the only price of the contract for which there is no arbitrage opportunity is $X_0 = V^h(0)$.

In real market practice the forward contracts are exchanged without any payments at time 0. Both parties of the contract negotiate over the price K that is specified in the contract. Thus the contract will be fairly priced if $S_1(0)-\frac{K}{1+r}=0$. As one can easily calculate, the delivery price should be thus

$$K = (1+r)S_1(0). \tag{7}$$

The equation (7) defines a **fair forward price** for an underlying asset S_1.

Based on Example 5 we can formulate a more general method of pricing, known as "the Law of One Price". Generally, it states that if two financial instruments bring the same payoffs at the end of investment period, then their initial prices should also be the same. Otherwise there is an arbitrage opportunity in the market. Here we state this law in the context of derivative instrument pricing.

Law of One Price. If a portfolio h replicates an attainable contingent claim X, then there is a unique fair price for X at the moment 0, which is equal to the initial value of the portfolio h:

$$X_0 = V^h(0) = h \cdot S(0).$$

Example 4 – cont.

Consider the derivative instrument X^1–the call option on the first asset with the strike price of 50. We will try to find a replication portfolio for this contingent claim. If we consider the model presented in Example 1, the definition of replication portfolio, given in equation (6), leads to the following system of linear equations:

$$\begin{aligned} 1.05h_0 + 47.25h_1 + 162.75h_2 &= 0, \\ 1.05h_0 + 47.25h_1 + 157.5h_2 &= 0, \\ 1.05h_0 + 63h_1 + 152.25h_2 &= 13. \end{aligned} \tag{8}$$

Its solution is $h_0 = -37.14$, $h_1 = 0.8254$, $h_2 = 0$. Thus the derivative instrument is attainable and according to the Law of One Price its initial price should be

$$X_0^1 = -37.14 + 0.8254 \cdot 50 + 0 \cdot 150 = 4.13.$$

Note that the replicating portfolio does not contain the second security, S_2.

Consider the same derivative instrument in the model from Example 3. Searching for a replicating portfolio leads us to the system of equations (8) with one additional equation:

$$1.05h_0 + 68.25h_1 + 147h_2 = 18.25. \tag{9}$$

As one can easily check, the new system does not have a solution, as the only solution of (8), i.e. $h = (-37.14, 0.8254, 0)$ does not fulfill the equation (9). Thus the contingent claim X^1 is not attainable in the model from Example 3.

Exercise 1. Find replicating portfolios for derivative instruments X^2 and X^3 from Example 4, assuming the model from Example 1.

Exercise 2. Show that a contingent claim in the model from Example 3 is attainable, only if its payoffs fulfill the condition $2X(\omega_1) - X(\omega_2) - 4X(\omega_3) + 3X(\omega_4) = 0$. A hint: consider the system of equations (8) with RHS (right hand side) defined by $X(\omega_1), \ldots, X(\omega_3)$. Find its general solution and find the value of $X(\omega_4)$ for which equation (9) is fulfilled.

The Law of One Price is applicable only to derivatives that can be replicated. It would be advantageous if we knew the simple criterion that allows us to check if the derivative instrument is attainable or not. Such a criterion exists, and indeed is a very simple one, for a special class of models. We call a model of market **complete** (or call the market itself complete) if all contingent claims are attainable.

Later we will establish a criterion to check if a market is complete. For now let us make the following observation. In the market model with *M* possible states of the world any contingent claim *X* has *M* possible payoffs in different states of the world. Finding a replicating portfolio requires solving a system of *M* linear equations:

$$S_0(1,\omega_1)h_0 + S_1(1,\omega_1)h_1 + \ldots + S_N(1,\omega_1)h_N = X(\omega_1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_0(1,\omega_M)h_0 + S_1(1,\omega_M)h_1 + \ldots + S_N(1,\omega_M)h_N = X(\omega_M).$$

The system of equation has a solution only if the number of variables is greater or equal to the number of equations. This gives us a "role of thumb" for checking completeness. If the number of independent assets in the model is smaller than the number of states of the world, then the model is incomplete. That is why the model from Example 3 is incomplete. It has four possible states of the world and only three securities.

3.1.4. Martingale pricing

We will describe another approach to pricing of derivative instruments. We assume that there is no arbitrage opportunity in the model, and thus, according to the First Fundamental Theorem, there is a martingale measure *Q*. Consider an attainable contingent claim X with the replicating portfolio *h*. According to the

Law of One Price, the initial price of X should be equal to the initial value of the portfolio h. We can perform the following calculations:

$$X_0 = V_h(0) = \sum_{n=0}^{N} h_n S_n(0) = \sum_{n=0}^{N} h_n E^Q\left[\beta(T) S_n(T)\right] =$$

$$= E^Q\left[\sum_{n=0}^{N} h_n \beta(T) S_n(T)\right] = E^Q\left[\beta(T) V^h(T)\right] = E^Q\left[\beta(T) X\right].$$

In this way we have obtained the simple formula for the prices of derivative instrument.

> **Martingale Pricing**. If X is an attainable instrument, then its unique fair price at the moment $t = 0$ is
>
> $$X_0 = E^Q\left[\beta(T) X\right], \tag{10}$$
>
> where Q is any martingale measure.

In the Martingale Pricing formula the expectation is calculated with respect to a martingale measure Q. As we have seen in Example 3, there can be more such measures. Howwever, this is not a problem, since it does not matter which measure we choose. The expected value will be the same.

In deriving equation (10) we have taken the riskless asset S_0 as the numéraire. Let us now assume that the numéraire is S_m, which requires that $S_m(t) > 0$ for all t and all the possible states of the world. The model is arbitrage-free, thus there exists a martingale measure Q^m for S_m as the numéraire. Let us take any attainable derivative instrument X and calculate the fair price X_0 for this instrument:

$$X_0 = V_0^h = \sum_{n=0}^{N} h_n S_n(0) = S_m(0) \sum_{n=0}^{N} h_n \frac{S_n(0)}{S_m(0)} = S_m(0) \sum_{n=0}^{N} h_n E^m\left[\frac{S_n(T)}{S_m(T)}\right] =$$

$$= S_m(0) E^m\left[\sum_{n=0}^{N} \frac{h_n S_n(T)}{S_m(T)}\right] = S_m(0) E^m\left[\frac{V^h(T)}{S_m(T)}\right] = S_m(0) E^m\left[\frac{X}{S_m(T)}\right].$$

We have obtained in this way another formulation of the Martingale Pricing rule.

Martingale Pricing. If X is an attainable instrument, then its unique fair price fulfills the following condition:

$$\frac{X_0}{S_m(0)} = E^m\left[\frac{X}{S_m(T)}\right], \tag{11}$$

where the expectation E^m is calculated with respect to martingale measure Q^m for the numéraire S_m.

Example 4 – cont.

Let us consider again the pricing of the derivative instrument X^1 (a call option on the first asset) in the mode1 from example 1. The model is free of arbitrage and its unique martingale measure is $Q(\omega_1)=Q(\omega_2)=Q(\omega_3)=\frac{1}{3}$. Using the equation (10) we can calculate the fair price of X^1:

$$X_0^1 = \frac{1}{3}\frac{0}{1.05} + \frac{1}{3}\frac{0}{1.05} + \frac{1}{3}\frac{13}{1.05} = 4.13.$$

We can also use the first risky asset, S_1, as the numéraire. The martingale measure for this numéraire is $Q(\omega_1)=Q(\omega_2)=0.3$, $Q(\omega_3)=0.4$. Thus

$$\frac{X_0^1}{50} = 0.3\cdot\frac{0}{47.25} + 0.3\cdot\frac{0}{47.25} + 0.4\cdot\frac{13}{63} = 0.08254.$$

The fair price of the option equals $X_0^1 = 50\cdot 0.08254 = 4.13$.

Exercise 3. Calculate the fair prices of the contingent claims X^2 and X^3 from Example 4. Assume that the model from Example 1 is true. Make calculations using S_0 and S_1 as the numéraire.

Martingale Pricing rule is applicable only to attainable contingent claims and it gives no information how to price a non-attainable derivative instrument in the market model without arbitrage opportunity. As it turns out, one cannot set a unique fair price for a contingent claim that is not attainable. However, it is possible to establish a range of possible prices that do not allow for arbitrage. One can do this by considering portfolios that give good approximation to the payoffs of the instrument.

Let us consider a contingent claim *X*. A portfolio *h* **super-replicates** *X* if its final value is no lower than the payoffs of *X* in all possible states of the world, i.e. $V^h(T) \geq X$. A portfolio that super-replicates a contingent claim *X* provides enough money to secure possible payoffs of *X*. Thus if the price of derivative instrument is higher than the initial value of super-replicating portfolio, there exists an opportunity for arbitrage. One can sell the contingent claim *X* and create the super-replicating portfolio *h*. The money from the portfolio will cover the payoffs connected with *X*. The difference between the price of the contingent claim and the value of the super-replicating portfolio is a riskless gain for the investor. Thus an initial value of any super-replicating portfolio is an upper bound on the price of the derivative claim.

On the other hand, we say that a portfolio *h* **sub-replicates** a contingent claim *X* if its final value never exceeds the payoffs of *X*, i.e. $V^h(T) \leq X$. Using argumentation analogous to the one from the previous paragraph, we can show that the fair price of the contingent claim cannot be lower than the initial value of a sub-replication portfolio. Sub-replicating portfolios set a lower bounds on the possible prices of a contingent claim.

Super- and sub-replicating portfolios set an interval for possible fair prices of a contingent claim. The price should not exceed the initial value of the cheapest super-replicating portfolio, and, on the other hand, it should be no lower than the initial value of the most expensive one from sub-replicating portfolios. As it turns out, the limitations on the fair price for a contingent claim can be also expressed using martingale measures.

The bounds on prices for a contingent claim. The fair price of any derivative instrument X should lie in the interval $\left[X_0^b, X_0^a\right]$, where X_0^b is the **bid price** or **buyer's price**, defined as

$$X_0^b = \max\{V_0^h : h \text{ sub-replicates } X\},$$

and X_0^a is the **ask price** or **seller's price**:

$$X_0^a = \min\{V_0^h : h \text{ super-replicates } X\}.$$

The bid and ask prices can be also calculated with the following formulae:

$$X_0^a = \max_{Q \in \mathbf{Q}} E^Q[\beta X], \tag{12}$$

$$X_0^b = \min_{Q \in \mathbf{Q}} E^Q[\beta X]. \tag{13}$$

where Q is the set of all martingale measures.

Example 4 – cont.

Consider the contingent claim X^1 (a call option on the first security) in the model presented in Example 1. As we have shown, the model is not complete and the derivative instrument is not attainable. Thus there is no unique fair price for this instrument. However, we can set bounds on possible prices, using formulae (12) and (13). The set of all risk-neutral measures is given by

$$\mathbf{Q} = \left\{ \left(q, \frac{1}{2} - \frac{1}{2}q, 1 - 2q, \frac{3}{2}q - \frac{1}{2} \right) : q \in \left(\frac{1}{3}, \frac{1}{2} \right) \right\}.$$

The expected value of discounted payoffs is

$$E^Q[\beta X] = 0 + 0 + \frac{13}{1.05}(1 - 2q) + \frac{18.25}{1.05}\left(\frac{3}{2}q - \frac{1}{2} \right) = \frac{55}{42}q + \frac{155}{42}$$

and it obtains minimal and maximal values for $q = \frac{1}{3}$ and $q = \frac{1}{2}$, respectively. The bid price equals $X_0^{1b} = \frac{55}{42} \cdot \frac{1}{3} + \frac{155}{42} = 4\frac{8}{63} = 4.13$ and the ask price is $X_0^{1a} = \frac{55}{42} \cdot \frac{1}{2} + \frac{155}{42} = 4\frac{29}{84} = 4.35$.

One can also use the fact that any martingale measure in this model can be expressed as a linear combination of two “extreme” measures Q_0 and Q_1. Taking expectation with respect to these measures, we obtain $E^0\left[\beta X^1\right] = \frac{1}{3} \cdot \frac{13}{1.05} = 4.13$ and $E^1\left[\beta X^1\right] = \frac{1}{4} \cdot \frac{18.25}{1.05} = 4.35$. The market price of the option X^1 should be between 4.13 and 4.35. Otherwise there is an arbitrage opportunity.

Exercise 4. Calculate the bounds on prices of the contingent claims X^2 and X^3 from Example 4. Assume that the model from example 3 is true.

3.1.5. Second Fundamental Theorem of Asset Pricing

Recall that a model is called complete if every contingent claim is attainable. At the end of section 3.1.3 we indicated that a market model can be complete only if the number of financial instruments is at least equal to the number of possible future states of the world. Here we express the condition for completeness of a model in terms of martingale measure.

If a contingent claim is attainable, it has a unique fair price. This means that its bid and ask prices are equal. According to the equations (12) and (13) such situation is possible only if the expected values of the discounted payoffs from the instrument are the same, regardless of the martingale measure that is used. On the other hand, if for a given contingent claim X the expected value of its discounted payoffs, $E^Q[\beta X]$, is the same for every martingale measure, then the bid price equals the ask price. This means that the cheapest super-replicating portfolio coincides with the most expensive sub-replicating portfolio, which in turn means that the contingent claim is attainable.

> Assume that the model is viable and let $\mathbf{Q}$ be the set of all martingale measures. A contingent claim X is attainable if and only if the expected value $E^Q[\beta X]$ is the same for any $Q \in \mathbf{Q}$.

If the set of all martingale measures consists of only one element, then the expected values of discounted payoffs from any instrument X, $E^Q[\beta X]$ are also one-element sets, which means that any derivative instrument is attainable. The model is complete. On the other hand, if there is more than one martingale measure, then there exists an instrument for which the expected values of discounted payoffs are different. Suppose, for example, that there are two different martingale measures Q_1 and Q_2. There is a state of the world, say ω_k, for which these two probabilities differ, i.e. $Q_1(\omega_k) \neq Q_2(\omega_k)$. Define a contingent claim X as follows. Let $X(\omega_k)=1/\beta$ and $X(\omega)=0$ for any other state of the world ω. Then we have $E^1[\beta X]=Q^1(\omega_k) \neq Q^2(\omega_k)=E^2[\beta X]$ and therefore the contingent claim X is not attainable. Thus the model is not complete. We have obtained the following result, known as the Second Fundamental Theorem.

Second Fundamental Theorem. Assume that the model is viable. The model is complete if and only if there exists only one martingale measure, i.e. the set **Q** is a singleton.

Considering again the models presented in examples 1 and 3, we can notice the difference between them. In the model from example 1 there is only one risk-neutral measure. Thus, according to the Second Fundamental Theorem, the model is complete. In the model from example 3 there are many (in fact–infinitely many) martingale measures. The model is viable but not arbitrage-free.

3.2. Pricing in multi-period models

A realistic model of financial markets should assume that there are more than two moments in which trade can take place. Here we extend the analysis from the previous subchapter to the situation in which there are many possible trading dates. We consider here only models with discrete and finite sets of trading dates and possible states of the world.

3.2.1. Multi-period stochastic model of market

We assume that between the initial date 0 and the terminal date T there are T periods of equal length. The trade on securities can take place only at the moments 0, 1, …, $T-1$, T. The planning horizon for an investor is thus $\mathbf{T}=\{0,1,\ldots,T-1,T\}$. There are $N+1$ assets and the security S_0 is a risk-free bond or bank account. This security is the default numéraire. Its initial value is $S_0(0)=1$ and in each period its price grows at a risk-free rate $r_t>-1$, which is known in advance at the moment $t-1$. The price of S_0 at the moment t equals

$$S_0(t)=(1+r_1)(1+r_2)\cdots(1+r_t).$$

In general, the risk-free interest rate can change randomly. If the riskless interest rate is constant, $r_t=r$, then the price of S_0 at the moment t equals $S_0(t)=(1+r)^t$. The discount factor β is defined as $\beta(t)=1/S_0(t)$. In case of a constant risk-free rate, the discount factor equals $\beta(t)=(1+r)^{-t}$.

The other securities are risky. The initial prices are known at time 0. The price $S_k(t)$ of stock k at time t is a random variable and its value is known only at time t and depends on the particular state of the world which appears on the market. The sequence of prices $S_k(0),\ S_k(1),\ \ldots,\ S_k(T)$ is a stochastic process. We write $S(t)=\left(S_0(t), S_1(t), \ldots, S_N(t)\right)^T$ to denote a stock price vector at moment t. The sequence of price vectors at the subsequent moments is a $(N+1)$-dimensional stochastic process. We also define a stochastic process that describe discounted prices $\tilde{S}(t)=\left(\tilde{S}_0(t), \tilde{S}_1(t), \ldots, \tilde{S}_N(t)\right)^T=\left(1, \beta(t)S_1(t), \ldots, \beta(t)S_N(t)\right)^T$.

The possible states of the world ω are complete scenarios of price changes up to the terminal date T. One can identify them with possible trajectories of prices. At any moment t an investor knows only partial history up to this moment. The information available to an investor at subsequent moments is described by a sequence of σ-algebras $F_0,\ F_1,\ \ldots,\ F_T$. Each σ-algebra F_t contains events that are known by the moment t. It is assumed that $F_0=\{\varnothing, \Omega\}$, i.e. at moment 0 we have no information about future changes in prices. The σ-algebra F_T consists of all subsets of the set Ω, which means that at the terminal date we know everything. We know exactly which state of the world was realized. For any moment t in the middle of the planning horizon sets in the σ-algebra F_t consist of these $\omega \in \Omega$ for which partial price history up to the moment t is the same. Thus, for example, F_1 contains all sets of scenarios for which prices at moment 1 are the same. The σ-algebra F_2 consists of the sets of scenarios in which prices at moments 1 and 2 are the same, etc.

Example 6

There are two periods. The risk-free rate is constant and equal to $r=0.05$. Thus the prices of the risk-free bond (default numéraire) are $S_0(0)=1$, $S_0(1)=1.05$, $S_0(2)=1.05^2=1.1025$. There is only one risky asset and its initial price is $S_1(0)=100$. At each moment the price of the stock can either rise by 20% or fall by 10%, with equal probability. Possible prices at moment 1 are $S_1(1)=120$ and $S_1(1)=90$. Possible prices at moment 2 are $S_1(2)=144$, $S_1(2)=108$ and $S_1(2)=81$. The possible trajectories of price changes are presented in Figure 1. Figure 2 presents possible evolution of discounted prices, i.e. prices divided by 1.05 at moment 1 and by $(1.05)^2$ at moment 2.

There are four possible scenarios, thus the sampling space is $\Omega=\{\omega_1, \omega_2, \omega_3, \omega_4\}$. At the initial moment 0 an investor knows nothing, so $F_0=\{\varnothing, \Omega\}$. At moment 1 an investor can distinguish between the scenario in which the price of the asset equals 120 (the set $\{\omega_1, \omega_2\}$) and the scenario with the price at moment 1 equalling 90 (the set $\{\omega_3, \omega_4\}$). The information at moment 1 is described by the σ-algebra $F_1=\left\{\varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\right\}$. At moment 2 the investor knows exactly which scenario has been realized. He knows the trajectory of price up to the final moment 2. The σ-algebra F_2 consists of all subsets of Ω.

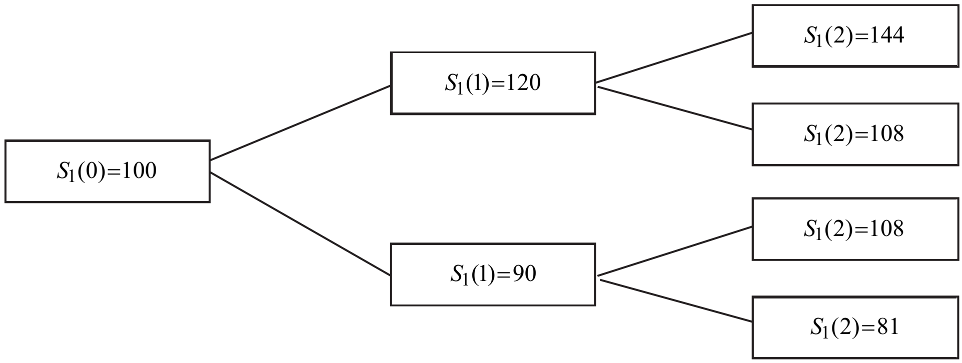

Figure 1. The model from Example 6–possible trajectories of prices

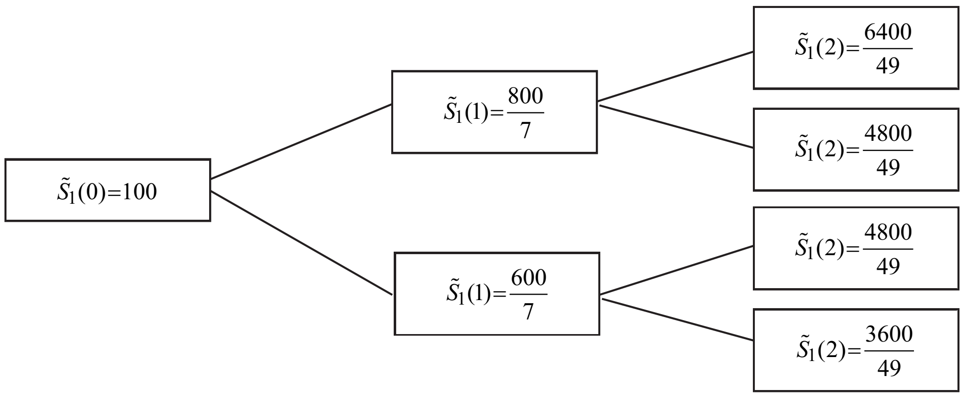

Figure 2. The model from Example 6–evolution of discounted prices

Example 7

We consider a market model with two periods. The risk-free rate is 5%. There are two risky securities on the market and their initial prices are $S_1(0)=100$ and $S_2(0)=50$. At moments 0 and 1 there are possible three types of price movements. The prices of stocks can grow. In this case the price of the first stock grows with rate 15% and of the second stock grows with rate 10%. The prices can fall – at rate of 20% for the first stock and at rate of 5% for the second stock. The third possibility is that the prices remain at the same level.

Possible states of the world are connected with trajectories of prices. For example, if the prices grow in both periods, then the trajectory of prices of risky securities is $S_1(1)=115$, $S_2(1)=55$, $S_1(2)=132.25$, $S_2(2)=60.5$. If the prices grow in the first period and remain constant in period 2, the trajectory of prices is $S_1(1)=115$, $S_2(1)=55$, $S_1(2)=115$, $S_2(2)=55$. At moment 1 only the prices at this moment and historical prices are known. Thus those two trajectories are indistinguishable.

Exercise 5. Draw all possible trajectories in the model from example 7. Check that the sample space in this model consists of 9 states of the world (possible paths of prices), i.e. $\Omega=\{\omega_1, \omega_2, \ldots, \omega_9\}$. Which states of the world are distinguishable at moment 1?

In the multi-period model an investor can adjust his portfolio at any moment of time. Moreover, in doing this he can account for changes in the current market situation. In a single-period model trading consists of buying a portfolio at moment 0 and selling it at the terminal moment *T*. In models with several periods an investor buys an initial portfolio at moment 0 and then is allowed to change it at intermediate moments $1, 2, \ldots, T-1$. At moment $t-1$ an investor chooses a portfolio $h(t)=(h_0(t), h_1(t), \ldots, h_N(t))^T$ and holds it until the next trading moment $t+1$. The sequence of portfolios forms a stochastic process–an investor does not know at the initial moment what portfolios he will buy at the next time steps. The stochastic process is predictable, i.e. the value $h(t)$ is known at moment $t-1$. Thus we come to the following definitions.

A **trading strategy** *h* is an $(N+1)$-dimensional predictable stochastic process describing the holdings of the investor from moment $t-1$ to moment *t*. We also define $h(0)=h(1)$. The **value process** of the trading strategy at moment *t* is given by

$$V^h(t)=h(t)\cdot S(t)=\sum_{n=0}^{N} h_n(t)S_n(t).$$

Notice that the value $V^h(t)$ is the value of the portfolio transferred from moment $t-1$, i.e. the value before the trading at moment *t*. The value of the trading strategy *h* after the trading at moment *t* equals

$$V_+^h(t)=h(t+1)\cdot S(t)=\sum_{n=0}^{N} h_n(t+1)S_n(t).$$

The **gain process** from the trading strategy h is defined as

$$G^h(t)=\sum_{s=1}^{t} h(s)\cdot\Delta S(s)=\sum_{s=1}^{t}\sum_{n=0}^{N} h_n(s)\Delta S_n(s),$$

where $\Delta S_n(t)=S_n(t)-S_n(t-1)$ and $\Delta S(t)=(\Delta S_0(t), \ldots, \Delta S_N(t))^T$. We define also the **discounted value** of the trading strategy – i.e. the value measured in the units of the numéraire:

$$\tilde{V}^h(t)=\beta(t)V^h(t)=h(t)\cdot\tilde{S}(t)=\sum_{n=0}^{N}h_n(t)\tilde{S}_n(t).$$

The discounted value after the trading at moment t equals

$$\tilde{V}_+^h(t)=\beta(t)V_+^h(t)=h(t+1)\cdot\tilde{S}(t)=\sum_{n=0}^{N}h_n(t+1)\tilde{S}_n(t).$$

The **discounted gain process** is defined as

$$\tilde{G}^h(t)=\sum_{s=1}^{t}h(s)\cdot\Delta\tilde{S}(s)=\sum_{s=1}^{t}\sum_{n=0}^{N}h_n(s)\Delta\tilde{S}_n(s),$$

where $\Delta\tilde{S}_n(t)=\tilde{S}_n(t)-\tilde{S}_n(t-1)$.

We consider a class of such trading strategies in which an investor put his own money only at the initial moment and later he does not add or withdraw any funds. All changes in investor's wealth are the results of changes of the prices in the market. For such strategies at any moment of time t the value of the strategy before and after transactions is equal. We call an investment strategy **self--financing** if for any moment t the following condition is fulfilled:

$$V_+^h(t)=V^h(t). \tag{14}$$

The value of a self-financing trading strategy changes only due to a profit or loss on investments. By simple bookkeeping calculations it can be shown that a trading strategy is self-financing if and only if the following condition is met:

$$V^h(t)=V^h(0)+G^h(t). \tag{15}$$

The condition can be also expressed with the use of discounted processes. A trading strategy is self-financing if and only if

$$\tilde{V}^h(t)=\tilde{V}^h(0)+\tilde{G}^h(t). \tag{16}$$

Exercise 6. Prove that equation (15) is a consequence of (14).

Exercise 7. Prove that equations (15) and (16) are equivalent.

Example 6 – cont.

Consider again the model from Example 6. Assume that an investor at the initial moment buys two stocks and takes a loan to finance it. His initial portfolio is $h(1)=(-200,2)$ and its value at the moment 0 equals $V^h(0)=0$. At moment 1 the money debt rises to 210. The value of portfolio before the trading takes place equals either $V^h(1)=240-210=30$, if the stock price has risen to 120 (i.e. in the states of the world ω_1 and ω_2), or $V^h(1)=180-210=-30$. Assume also that the investor has decided at $t=0$ that if stock price rises, he will buy an additional share, and if it falls, he will sell one share. If his strategy is to be self-financing, then in case of the rise it should fulfill the condition $1.05h_0(2)+360=30$, and $h_0(2)=-2200/7$. The portfolio created at moment 1 for the states of the world ω_1 and ω_2 is $h(2)=\left(-\frac{2200}{7},3\right)$. If the price falls, the portfolio should fulfill the condition $1.05h_0(2)+90=-30$, thus $h_0(2)=-800/7$ and the portfolio created at moment 1 is $h(2)=\left(-\frac{800}{7},1\right)$.

The final value of the strategy h depends on the state of the world:

$$V^h(2,\omega_1)=-\frac{2200}{7}(1.05)^2+3\cdot 144=85.5,$$

$$V^h(2,\omega_2)=-\frac{2200}{7}(1.05)^2+3\cdot 108=-22.5,$$

$$V^h(2,\omega_3)=-\frac{800}{7}(1.05)^2+1\cdot 108=-18,$$

$$V^h(2,\omega_4)=-\frac{800}{7}(1.05)^2+1\cdot 81=-45.$$

3.2.2. No-arbitrage and martingale measures

Here we consider an idea of arbitrage in a model with multiple time periods. The main concept is the same as in single-period models: an arbitrage is an opportunity to obtain profits without bearing any risk. Of course, in a realistic model of financial market, there should be no possibility for such strategies. In contrary to single period models, we define an arbitrage opportunity rather in terms of strategies then single portfolios.

An investment strategy h is an **arbitrage opportunity** if it is self-financing, its initial value is zero, $V^h(0)=0$ and its value at terminal date satisfies:

$$P\left(V^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(V^h(T)>0\right)>0.$$

If in a model such a strategy does not exist, then the model is **arbitrage--free** or **viable**.

As in the case of a single-period model, the existence of an arbitrage opportunity can be expressed also using a discounted value process or a discounted gain process.

There exists an arbitrage opportunity in the model, if and only if there exists a self-financing trading strategy h such that $\tilde{V}^h(0)=0$ and at the terminal data its discounted value fulfills:

$$P\left(\tilde{V}^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{V}^h(T)>0\right)>0$$

or there is a portfolio h such that

$$P\left(\tilde{G}^h\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{G}^h>0\right)>0.$$

As we have seen in case of single-period models, there is a close connection between viability of the market and risk-neutral measures. Similar connection exists in a multi-period model. However, now the meaning of a risk-neutral measure has changed. In the multi-period models it concerns stochastic processes.

A probabilistic measure Q is called a **martingale measure** (or **risk-neutral measure**) if the discounted prices of all the instruments are martingales, i.e. their current discounted prices are equal to expected values of future discounted prices

$$\tilde{S}_n(s)=E^Q\left[\tilde{S}_n(t)\,|\,F_s\right]=E^Q\left[\beta(t)S_n(t)\,|\,F_s\right] \tag{17}$$

for any $s<t$.

The following theorem is an extension of First Fundamental Theorem of Asset Pricing for models with multiple periods. As in the case of single-period models, viability of a model is equivalent to the existence of a martingale measure. In a viable model current discounted prices are expected values of future (discounted) prices, albeit the expectations can be calculated with respect to some artificial probability measure.

> **First Fundamental Theorem**. The model of financial market is arbitrage-free if and only if there exists a martingale measure.

Example 6 – cont.

Let us check if the model presented in Example 6 is complete. The dynamics of discounted prices is presented in Figure 2. Let us begin with the prices at moment $t=1$. In the set $\{\omega_1, \omega_2\}$ discounted price $\tilde{S}_1(1)=\frac{800}{7}$ should be equal to expected values of discounted prices at moment $t=2$ in the states of the world ω_1 and ω_2. Denote by $q_{12,1}$ and $q_{12,2}$ the conditional probabilities that if at moment 1 the scenario $\{\omega_1, \omega_2\}$ is being realized, then at moment 2 it will end in the state of the world ω_1 or ω_2, respectively. If the pair $q_{12,1}$ and $q_{12,2}$ defines martingale probability, then the following equation must be fulfilled:

$$\frac{800}{7}=q_{12,1}\frac{6400}{49}+q_{12,2}\frac{4800}{49}.$$

Together with the condition that probabilities should sum up to 1 (i.e. $q_{12,1}+q_{12,2}=1$), it gives a system of linear equations, whose solution is $q_{12,1}=q_{12,2}=\frac{1}{2}$. Consider now the set $\{\omega_3, \omega_4\}$ and denote by $q_{34,3}$ and $q_{34,4}$ the probabilities that the state of the world ω_3 (or ω_4, respectively) will be realized under the condition that at moment 1 the current scenario is $\{\omega_3, \omega_4\}$. For a martingale measure the following system of linear equation should be fulfilled:

$$\frac{600}{7}=q_{34,3}\frac{4800}{49}+q_{34,4}\frac{3600}{49},$$

$$q_{34,3}+q_{34,4}=1.$$

The unique solution is $q_{34,3}=q_{34,4}=\frac{1}{2}$.

Let us consider now the initial moment. The initial price is $S_1(0)=100$. Denote by $q_{0,12}$ and $q_{0,34}$ the probabilities that at moment $t=1$ the price will be 120 or 90, respectively. For a risk-neutral measure the following system of equations should hold:

$$100=q_{0,12}\frac{800}{7}+q_{0,34}\frac{600}{7},$$

$$q_{0,12}+q_{0,34}=1.$$

Its unique solution is again $q_{0,12}=q_{0,34}=\frac{1}{2}$.

The martingale probabilities are given as multiplications of conditional probabilities along the trajectory connected with a specific state of the world. In this model there exists a unique martingale measure Q, given by the following probabilities:

$$Q(\omega_1)=q_{0,12}q_{12,1}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4},\quad Q(\omega_2)=q_{0,12}q_{12,2}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4},$$

$$Q(\omega_3)=q_{0,34}q_{34,3}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4},\quad Q(\omega_4)=q_{0,34}q_{34,4}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}.$$

According to the First Fundamental Theorem the model is viable.

Example 7 – cont.

In the model in this example there are 9 possible states of the world, i.e. $\Omega=\{\omega_1,\ldots,\omega_9\}$, (check the solution of Exercise 5). Martingale probabilities can be calculated similarly as in Example 6–by considering conditional probabilities of changes from one moment to the next. For example, if there was an increase in prices in the first period and at moment 1 the stock prices are $S_1(1)=115$ and $S_2(1)=55$ (the set of scenarios $\{\omega_1,\omega_2,\omega_3\}$), then at the final moment there are possible three schemes of prices: $S(2)=(132.25,60.5)$ (the scenario ω_1), $S(2)=(115,55)$ (the scenario ω_2) and $S(2)=(92,52.25)$ (the scenario ω_3). Denote by q_1, q_2 and q_3 the (conditional) probabilities of these price movements. For the martingale measure the following system of linear equations should be fulfilled:

$$\frac{115}{1.05}=q_1\frac{132.25}{(1.05)^2}+q_2\frac{115}{(1.05)^2}+q_3\frac{92}{(1.05)^2},$$

$$\frac{55}{1.05} = q_1 \frac{60.5}{(1.05)^2} + q_2 \frac{55}{(1.05)^2} + q_3 \frac{52.25}{(1.05)^2},$$

$$q_1 + q_2 + q_3 = 1.$$

One can check that the only solution is $q_1 = 0.6,\ q_2 = q_3 = 0.2$. Similarly, one can calculate conditional martingale probabilities for the rest of possible scenarios. The martingale probabilities can be calculated by multiplying conditional probabilities along each possible path.

Exercise 8. Calculate the rest of martingale conditional probabilities in Example 7 and check that martingale probabilities are as follows: $Q(\omega_1) = 0.36$, $Q(\omega_2) = Q(\omega_3) = Q(\omega_4) = Q(\omega_7) = 0.12$ and $Q(\omega_5) = Q(\omega_6) = Q(\omega_8) = Q(\omega_9) = 0.04$.

3.2.3. Replication and market completeness

A notion of a contingent claim has its counterpart in multi-period models. As the name suggests, it is a financial instrument whose payoffs depend on the condition in the market: on current prices of assets or on their history. Here we consider only the simplest kind contingent claims, which brings payoffs only at a fixed date in the future. Without losing generality we can assume that the payment date is the final moment in a model *T*. Such financial instruments are called European contingent claims (or European derivatives), to distinguish them from American derivatives, where payoffs can take place in any time, chosen by the owner of such an instrument. The payoff of a contingent claim depends on the complete history of price changes, i.e. on the scenario ω, which was realized. Hence the following definition:

A **European derivative** (or **contingent claim**) with expiry date *T* is a random variable $X : \Omega \to R$.
The value $X(\omega)$ for any scenario ω represents payoff to the owner of at moment *T* in case the scenario ω was realized.

Hence we deal with European derivatives only, we will use terms "derivative" or "contingent claim", without specifying that it is a European one.

The notion of replication also naturally extends to the case of multi-period models. If there exists a self-financing strategy *h*, such that its final value equals payoffs of a derivative in every scenario, i.e.

$$V^h(T,\omega)=X(\omega) \text{ for all } \omega\in\Omega,$$

we say that the contingent claim X is **attainable** and the strategy h **replicates** it (is a replicating strategy).

In the previous definition it is important that a replicating strategy should be self-financing. If it is so, then a derivative and a replicating strategy are financially equivalent. In both cases (either buying a derivative instrument or creating a replicating strategy) an investor has to make use of his own money only at the initial moment, and at the final date he will obtain payoff connected with the derivative instrument or its equivalent. If the strategy is not self-financing, then it differs from derivative instrument in additional payments or withdrawals.

Using the same arbitrage argument as in the case of single-period models, we came to the conclusion that the fair price of any attainable contingent claim should be equal to the current value of a strategy that replicates this claim. Hence we can state the following pricing rule:

Non-Arbitrage Pricing. If a derivative instrument X is attainable and h is its replicating strategy $\left(V^h(T)=X\right)$, then the fair price of X at the initial moment equals the initial value of the replicating strategy:

$$X_0=V^h(0)\,. \tag{18}$$

Moreover, if we denote by X_t the fair price of the derivative instrument X at any moment $t\in\{0,1,\ldots,T\}$, then it should be equal to the value of strategy h:

$$X_t=V^h(t). \tag{19}$$

Example 6 – cont.

In the model from Example 6 let us consider a European call option on stock with the execution price of 100 and execution date 2. The payoff of the derivative instrument is defined as $X=\left(S_1(2)-100\right)^+$. Thus $X\left(\omega_1\right)=44$, $X\left(\omega_2\right)=X\left(\omega_3\right)=8$ and $X\left(\omega_4\right)=0$. We will calculate a replicating strategy. Let us start with moment 1 and assume that scenario $\{\omega_1,\omega_2\}$ is being realized (see Figure 1). An investor creates a portfolio (h_0,h_1), whose value final value should be equal to 44 (on ω_1) or 8 (on ω_2). This gives the following system of equations:

$$(1.05)^2\,h_0+144h_1=44,$$

$$(1.05)^2 h_0 + 108 h_1 = 8,$$

whose unique solution is $h_0 = -\frac{100}{(1.05)^2}$, $h_1 = 1$. The value of this portfolio at moment $t = 1$ equals $V^h(1, \{\omega_1, \omega_2\}) = -\frac{100}{(1.05)^2} \cdot 1.05 + 1 \cdot 120 = \frac{520}{21} \approx 24.76.$

Consider now the set of scenarios $\{\omega_3, \omega_4\}$ at moment $t = 1$. We obtain the following system of equations:

$$(1.05)^2 h_0 + 108 h_1 = 8,$$

$$(1.05)^2 h_0 + 81 h_1 = 0,$$

whose unique solution is $h_0 = -\frac{24}{(1.05)^2}$, $h_1 = \frac{8}{27}$. The initial value of the portfolio is $V^h(1, \{\omega_3, \omega_4\}) = -\frac{24}{(1.05)^2} \cdot 1.05 + \frac{8}{27} \cdot 90 = \frac{80}{21} \approx 3.81.$

The investor at the initial moment creates a portfolio (h_0, h_1). As the replicating strategy is self-financing, the value of this portfolio at moment 1 should be equal to either $V^h(1, \{\omega_1, \omega_2\})$ or $V^h(1, \{\omega_3, \omega_4\})$. This gives the following system of equations:

$$1.05 h_0 + 120 h_1 = \frac{520}{21},$$

$$1.05 h_0 + 90 h_1 = \frac{80}{21}.$$

The solution is $h_0 = -\frac{24800}{21^2}$, $h_1 = \frac{44}{63}$ and the initial value of the replicating strategy equals $V^h(0) = -\frac{24800}{21^2} + \frac{44}{63} \cdot 100 = \frac{6000}{21^2} \approx 13.61$.

The fair price of an option at moment 0 is 13.61. The price at moment 1 should be equal either to 24.76 or 3.81–depending on whether the price of the stock rises or falls.

Formulae (18) and (19) allow to find the fair price of any attainable derivative instrument. As in the single-period case, we call a model of market **complete,** if all contingent claims are attainable. The Second Fundamental Theorem naturally extends to models with many periods.

Second Fundamental Theorem. Assume that the model is viable. The model is complete if and only if there exists only one martingale measure.

Example 8

Let us take the model from Example 6 and assume that in the second period there is also a possibility that stock price will not change (but only if there is growth in the first period). The model is presented in Figure 3. There are five possible scenarios (of states of the world): $\omega_1, \ldots, \omega_5$. It can be shown that in this model there are infinitely many martingale measures: $Q(\omega_1)=\frac{q}{2}$, $Q(\omega_2)=\frac{3}{4}-\frac{3q}{2}$, $Q(\omega_3)=q-\frac{1}{4}$, $Q(\omega_4)=Q(\omega_5)=\frac{1}{4}$, where q is any number between $\frac{1}{4}$ and $\frac{1}{2}$. Thus the model is incomplete.

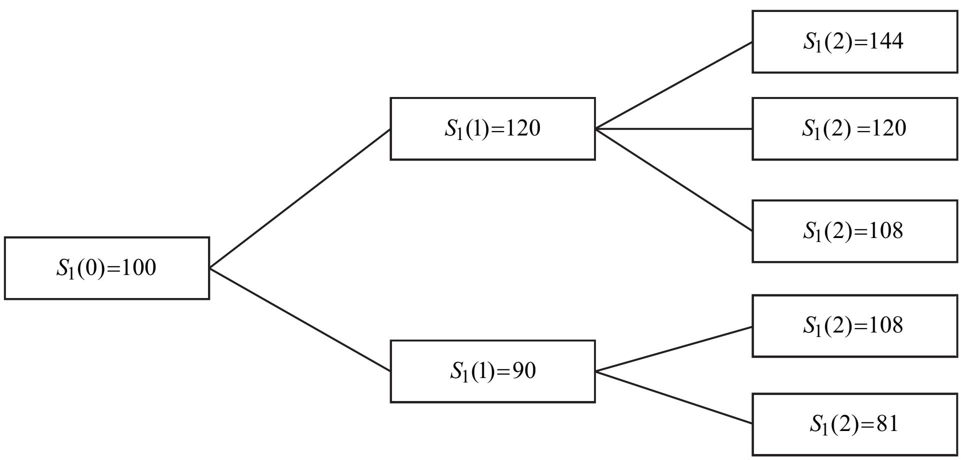

Figure 3. The model from Example 8

Exercise 9. Calculate all the martingale measures in the model from example 8. Check that they have a from presented in this example. Check directly that model is not complete (find any unattainable contingent claim).

The equations (18) and (19) present one approach to the pricing of derivative instruments. There is also another one, which takes advantage of the fact, that in a viable market discounted prices of all the assets are martingales (under a martingale measure). Repeating the reasoning from the beginning of section 3.1.4, we can state the following pricing rule:

Martingale Pricing. The discounted fair price of any attainable derivative instrument X is a martingale under any martingale measure Q:

$$\beta(s)X_s = E^Q\left[\beta(t)X_t \mid F_s\right], \quad \text{for any } s < t. \tag{20}$$

In particular, the initial price of X equals

$$X_0 = E^Q[\beta(T)X]. \tag{21}$$

Example 6 – cont.

Let us compute again the fair price of the European call option with the strike price of 100, using the martingale pricing rule. We know that the derivative is attainable, as the model is complete (there is only one martingale measure). According to equation (21) the initial price equals

$$X_0 = \frac{1}{4}\cdot\frac{44}{(1.05)^2} + \frac{1}{4}\cdot\frac{8}{(1.05)^2} + \frac{1}{4}\cdot\frac{8}{(1.05)^2} + \frac{1}{4}\cdot\frac{0}{(1.05)^2} = \frac{60}{4\cdot(1.05)^2} \approx 13.61.$$

With equation (20) we can compute prices at moment 1. For the set of scenarios $\{\omega_1, \omega_2\}$ we have

$$\frac{X_1}{1.05} = \frac{1}{2}\cdot\frac{44}{(1.05)^2} + \frac{1}{2}\cdot\frac{8}{(1.05)^2} = \frac{52}{2\cdot(1.05)^2}, \quad \text{thus} \quad X_1 = \frac{52}{2.1} \approx 24.76,$$

while for the set of scenarios $\{\omega_3, \omega_4\}$:

$$\frac{X_1}{1.05} = \frac{1}{2}\cdot\frac{8}{(1.05)^2} + \frac{1}{2}\cdot\frac{0}{(1.05)^2} = \frac{8}{2\cdot(1.05)^2}, \quad \text{thus} \quad X_1 = \frac{8}{2.1} \approx 3.81.$$

Exercise 10. In the model from Example 7 consider an option which gives its owner the right to buy at moment 2 a portfolio consisting of one share of first asset and two shares of second asset at the price of 190. The payoffs of the derivative instrument are thus $X = \left(S_1(2) + 2S_2(2) - 190\right)^+$. Calculate the fair price of this option and find a replicating strategy.

Exercise 11. In the model from Example 6 consider an Asian call option with the strike price of 95. The payoff of such an option depends not on the price of the underlying asset in the final moment but on the average price during the whole period till the exercise date, i.e. the payoff is $X = \left(\overline{S} - 95\right)^+$, where $\overline{S} = \frac{1}{3}\left(S_1(1) + S_1(2) + S_1(3)\right)$. Find the fair price of the option and calculate replicating strategy.

Exercise 12. The payoff function of a European call option is $X^c = \left(S(T) - K\right)^+$, where $S(T)$ is the price of the underlying asset S at the exercise moment T and K is the strike price. The payoff of a put option with the same strike price and exercise date is $X^p = \left(K - S(T)\right)^+$. Let us denote by X_t^c and X_t^p the prices of call and put options at any moment $t < T$. Using the arbitrage argument show that the following call-put parity must hold:

$$X_t^c - X_t^p = S(t) - P(t,T)K,$$

where $P(t,T)$ is the discount factor from moment T to moment t (the value at t of 1 unit of money that will be paid at moment T).

Solutions to selected exercises

Exercise 1. Replicating portfolio for instrument X^2 is (10.4762,–0.1746,0). For instrument X^3 it is (–190.4762,1,1).

Exercise 3.

$$X_0^2 = E^Q\left[\beta(T)X^2\right] = \frac{1}{3}\cdot\frac{2.75}{1.05} + \frac{1}{3}\cdot\frac{2.75}{1.05} + \frac{1}{3}\cdot\frac{0}{1.05} = 1.75,$$

$$X_0^2 = S_1(0)E^1\left[\frac{X^2}{S_1(T)}\right] = 50\cdot\left[0.3\cdot\frac{2.75}{47.25} + 0.3\cdot\frac{2.75}{47.25} + 0.4\cdot\frac{0}{63}\right] = 1.75,$$

$$X_0^3 = E^Q\left[\beta(T)X^3\right] = \frac{1}{3}\cdot\frac{10}{1.05} + \frac{1}{3}\cdot\frac{4.75}{1.05} + \frac{1}{3}\cdot\frac{15.25}{1.05} = 9.52,$$

$$X_0^3 = S_1(0)E^1\left[\frac{X^3}{S_1(T)}\right] = 50\cdot\left[0.3\cdot\frac{10}{47.25} + 0.3\cdot\frac{4.75}{47.25} + 0.4\cdot\frac{15.25}{63}\right] = 9.52.$$

Exercise 4. The price of X^2 should be between 1.75 and 1.96. The price of X^3 equals 9.52. This instrument is attainable.

Exercise 6. We have

$$V^h(t) = \sum_{n=0}^{N} h_n(t)S_n(t) = \sum_{n=0}^{N} h_n(t)\left[S_n(t-1) + \Delta S_n(t)\right] = \sum_{n=0}^{N} h_n(t)S_n(t-1) +$$

$$+ \sum_{n=0}^{N} h_n(t)\Delta S_n(t) = V_+^h(t-1) + \Delta G^h(t) = V^h(t-1) + \Delta G^h(t),$$

where $\Delta G^h(t)=G^h(t)-G^h(t-1)$. Repeating this derivation for moments $t-1$, $t-2$, …, 2, 1 we obtain

$$V^h(t) = V^h(0) + \Delta G^h(1) + \Delta G^h(2) + \ldots + \Delta G^h(t) = V^h(0) + G^h(t).$$

Exercise 7. The derivation is analogous to the one in Exercise 6.

Exercise 9. Martingale measure: $Q(\omega_1)=\frac{q}{2}$, $Q(\omega_2)=\frac{3}{4}-\frac{3q}{2}$, $Q(\omega_3)=q-\frac{1}{4}$, $Q(\omega_4)=Q(\omega_5)=\frac{1}{4}$, where $q\in\left(\frac{1}{4},\frac{1}{2}\right)$. Attainable instruments should fulfill the condition $X(\omega_1)-3X(\omega_2)+2X(\omega_3)=0$, so for example the instrument which pays 1 in ω_1 and 0 in all other states of the world is not attainable.

Exercise 10. The payoffs of the instrument are 63.25 in ω_1, 35 in ω_2 and ω_4, 10 in ω_5 and 6.5 in ω_3 and ω_7. In ω_6, ω_8 and ω_9 the payoff is 0. The martingale measure is $Q(\omega_1)=0.36$, $Q(\omega_2)=Q(\omega_3)=Q(\omega_4)=Q(\omega_7)=0.12$, $Q(\omega_5)=Q(\omega_6)=Q(\omega_8)=Q(\omega_9)=0.04$. The price of the instrument at $t=0$ equals 30.05.

Exercise 11. The payoffs of the instrument are $X(\omega_1)=26\frac{1}{3}$, $X(\omega_2)=14\frac{1}{3}$, $X(\omega_3)=4\frac{1}{3}$, $X(\omega_4)=0$. The price at $t=0$ is 10.20.

Exercise 12. Consider a portfolio in which there is one unit of the underlying asset S, one call put option and $-P(t,T)K$ in money (in other way: there is a debt in the amount $P(t,T)K$ and until exercise moment T the value of the debt grows to K). The portfolio is created at moment t and held until moment T. Show that the final value of this portfolio equals the payoffs of the call option–regardless of the final price of the underlying asset. The portfolio replicates the call option, so its initial value should equal the initial price of this option.

Further readings

Textbook concerning basics of derivative instruments and pricing in discrete time

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CHAPTER 4
CORPORATE FINANCING–DESIGNING AND OFFERING SECURITIES

Securities are packages of cashflow and control rights
Stephen C. Myers
Handbook of the economics of finance

4.1. Designing securities

4.1.1. Control rights versus cash flow rights

Different kinds of securities issued by companies represent different claims of their holders that generally refer to **cash flow rights** and **control rights**. The scope of these rights is generally given by law but to some extent it may be limited or expanded in corporates' articles of incorporation or charters and bylaws.

Common stocks entitle their holders to **participate in issuer's profits**, usually in the form of a cash dividend (cash flow rights) and to decide about **how** the issuing **company should be managed (**control rights**)**. The latter is expressed by **voting rights**, which are usually (but not always) **proportional to the number of stocks** held. Shareholders holding the highest stakes, called **major shareholders** (or main shareholders) are usually able to appoint managers who share their vision of pursuing company's strategy or fire those who don't. It is obvious when they hold the **majority of votes** but even if they do not, and the ownership structure is dispersed, they will still be able to decide, because rarely all the voting rights are represented during shareholders' meetings where crucial decisions are usually made.

Bondholders have cash flow rights too, but their control rights are limited to the minimum. Nevertheless, their **claims** on cash flows are **superior** to the claims of common shareholders and other company's stakeholders. It means, among others, that in case of a liquidation they must be paid off before any shareholders (including also preferred shareholders).

Table 1. Basic features of main types of securities issued by corporations

Areas of comparison	Common stock	Preferred stock	Corporate bond
Voting rights	yes	limited (usually)	no
Cash flow rights–claim's character	residual	residual / defined	defined
Cash flow rights–form	dividends	dividends	interest and debt repayments
Cash flow rights–seniority	low	medium	high
Maturity	undefined	undefined	defined
Risk (investors' perspective)	high	medium	low
Risk (issuers' perspective)	low	medium	high

Claims of the preferred shareholders **differ across countries** but generally they represent a **mix** of claims of common shareholders and bondholders, that is why they are called hybrid securities. Typically their cash flow rights are stronger than common shareholders' ones but sometimes at the cost of control rights. This is the case in the U.S. where companies are **obliged to pay dividends to preferred shareholders**, similarly as they pay interest to debtholders, but, on the other hand, **preferred stocks usually give no voting rights** or these rights are much limited. That is why they are more similar to bonds than common stocks. On the contrary, in Poland preferred stocks are more similar to common shares. First of all, Polish companies are not obliged to pay dividends to preferred shareholders. They can do so if and only if a company generates profits (the same condition as for common stocks). Secondly, dividends for preferred shareholders are not set in advance in Poland, but on the other hand they can be higher than for common shareholders (up to 150% according to the current law). Polish preferred stocks differ from U.S. ones also in other aspects. They typically give extra voting rights (like dual-class shares in the U.S.). Nevertheless, Polish company law (Kodeks spółek handlowych or ksh) limits the maximum parity to two votes per one share. It is worth mentioning that before ksh was introduced in 2000 the limit was set much higher: five votes per one share. That is why many Polish companies (included listed companies) that were incorporated before 2000 still have preferred stocks giving their holders (typically founders) five votes per share.

Notice that common shareholders control the company but in practice the so-called **effective control** is **in hands of managers** (directors), especially when ownership structure is **dispersed,** which often leads to the **separation of ownership and control**. Such a situation is common in **public companies** and may cause the so-called **agency problems**, which stem from potential **conflicts of interest** between owners and managers acting as owners' agents. This issue is very important in both corporate finance and **corporate governance** but goes far beyond the scope of this chapter.

4.1.2. One share–one vote principle and control-enhancing mechanisms

Traditionally **one common stock gives one voting right**. It means that shareholders' control rights are **proportional** to their cash flow rights. The number of total votes equals the number of total shares. An investor willing to gain control over the company will have to bear proportional costs of acquiring required number of stock. The same refers to current shareholders, especially **company's founders** – to retain control they have to hold majority of shares.

For some reasons company's owners may be more interested in control rights than in cash flow rights, meaning that they are willing to sacrifice cash flows to retain control over the company. Such a situation is quite common in European companies (even listed companies) controlled by founders or founders' families. Sometimes–as in case of the so-called **golden share**–their holders do not even have to sacrifice anything, still controlling the company. Golden shares are typically held by states in state-controlled companies and give them the ultimate right to vote "for" or "against", outvoting other shareholders. It means that all the decisions have to gain the approval of the golden share's holder.

While the golden share is an extreme example of violating the one-share one-vote principle and it would be difficult to implement such a solution in a public company, there are many other mechanisms that can be used to increase some owners' control rights. These mechanisms called **control-enhancing mechanisms** (CEMs) or **proportionality-limiting measures** (PLMs) include among others:

- dual-class shares (multiple voting rights shares),
- pyramid structures and cross-shareholdings,
- voting restrictions (voting rights ceilings),
- disproportional board representation,
- holding period requirements,
- shareholders agreements.

Dual class shares may increase or decrease control rights of their holders. If dual class shares give more than one vote per share (e.g. 10 votes per share), the voting power of their holders is disproportionally higher than their cash flow rights. Such shares may be issued and offered to current shareholders if they want to increase their control rights. However, when companies want to raise more capital from outside investors, they can issue dual class shares with less than one vote per share (e.g. 1/10 votes per share). By doing it, the existing shareholders **reduce** the **new issue dilution effect** and can retain control over the company. Of course, if the number of newly issued shares equals the number of existing shares, future dividends will be divided proportionally between new and old shareholders. Dual class shares are **very popular in** many countries including **the U.S.** and **Scandinavian** countries. Listed companies may issue dual class shares which are traded on the open market just like any other shares. If two classes of shares are traded on the market, they are typically named class A shares and class B shares and the latter one usually carries less voting rights. In some countries, e.g. in continental Europe, including Poland, listed companies **cannot issue multiple voting rights shares** but they may still issue non-voting shares. Companies listed on Warsaw Stock Exchange have preferred stocks outstanding issued before these companies went public. However, they cannot be traded on the open market via stock exchange (according to the Polish stock exchange regulations they should be converted into common shares first).

In practice

Alphabet (Google) is a good example of multiple voting rights issuer. The company's stocks form a multi class structure: class A stocks give one vote, class B stocks give 10 votes and class C stocks are non-voting stocks. Class B shares are held by the founders and are not traded on the open market. Class A (GOOGL) and class C (GOOG) are publicly available as they are traded on NASDAQ stock exchange.

Pyramid structures, also called **cascading holdings**, enable the holder to control assets worth much more than the value of the controlling stake in the **ultimate parent company**. If you own 51% of common shares of an ultimate parent company, you also control, although however indirectly, all **lower-tier companies** within the holding structure.

Example 1

Pharaoh Inc. has one million shares outstanding that give one voting right per share, with a market price of $10. Investor K owns 60% of shares, thus her stake is worth $6m. Company finances its operations partly with debt. Its D/E ratio equals 1. It has 60% stakes in other companies *L-T*1 and *L-T*2 that have identical size and capital structure to the parent company. What is the value of assets controlled by investor *K?*

Pharaoh's equity value: $10 · 1m = **$10m**

Pharaoh's debt value = $10m · 1 = **$10m**

Pharaoh's total firm value = **$10m + $10m = $20m**

Value of Pharaoh stakes in *L-T*1 and *L-T*2 (separately): 60% · $10m = **$6m**

Value of *L-T*1's (and *L-T*2's) assets in excess of Pharaoh's stake = $20m + – $6m = $14m

Total assets controlled by investor *K:* $20m + $14m + $14m = **$48m**

Notice, that each tier in a holding structure multiplies the assets controlled by the major shareholder of the ultimate parent company.

Multiple voting class shares and pyramid structures are most commonly used control-enhancing mechanisms but some companies use also other mechanisms that are included in their corporates' articles of incorporation or charters. A good example is a **voting rights ceiling** meaning that shareholders can exercise a limited number of votes regardless of the number of shares they hold. Founders of some companies are **allowed to appoint a certain number of board members** (managers) regardless of the number of shares they hold. This privilege typically expires when founders' stake goes below a minimum required (e.g. 10%). In some countries investors need to hold shares for a minimum required holding period (e.g. one year) if they want to exercise their voting rights. This restriction usually refers to shares with special voting rights, not to the common shares.

4.1.3. Seniority and security in corporate debt instruments

Equities have different control rights and cash flow rights than debt instruments, but it is worth mentioning that securities within each group of securities may also differ among themselves. This may refer to stocks (common shares, preferred shares and multiple voting rights shares) but it mostly refers to corporate bonds, which can have very different structures defined separately in bond contracts each time when new bonds are issued.

Corporate bonds may differ among themselves in terms of **seniority** and **collateral** used to secure bondholder's claims. Seniority refers to the **power of claims** and the **priority** in their satisfaction in case of a bankruptcy. Collateral is typically **a property** or any other asset that secures debtholder's claim. It means that a company cannot sell the asset without permission of the debtholder and once it gets the approval, it has to **use the proceeds** from selling the asset to **satisfy debtholders' claims first**. There are some kinds of debt instruments issued by corporations that are secured (have collateral), e.g. **mortgage bonds,** but generally **most of corporate bonds are unsecured**–they are called **debentures**. Debenture holders' claims are secured only by the general **creditworthiness** of the issuer (which is assessed by **rating agencies**).

Because debentures are unsecured, their holders typically protect themselves in a different way. They put the so called **covenants** (restrictive clauses) **in a bond contract**. Covenants may limit to some extent issuer's control over the use of proceeds from the debt raised (as in case of bank loans), but also protect debtholders from the company's future actions that could impose a higher risk on their claims. The best example is a covenant that prohibits the company from raising any other debt without the approval of current debtholders. It is an extremely rigid clause which is usually softened: current debtholders may accept new debt issues but only if their claims remain **superior** to the claims of future bondholders. Thus their bonds are treated as **senior bonds** and any other bonds will be **junior bonds**. This is how **debt seniority structure** is formed.

Companies often issue many series of bonds, so bondholders holding bonds that were issued in different times may have claims of different seniority. If a bond A is **subordinated** to previously issued bond B (senior debt), it means that in case of bankruptcy, claims of bond A holders have priority over the claims of holders of bond B.

4.2. Aims of security offerings

Companies issue and offer securities first of all in order to **raise capital** necessary for financing future investment projects (e.g. capital expenditures, R&D). Companies raise funds basically **by issuing new shares** (increasing its equity) or **by issuing bonds** (increasing its liabilities). Of course, there are more ways used by companies to raise capital, which can come from many sources like bank loans or leasing agreements. Some companies also issue hybrid securities such as preferred stocks or convertible bonds that combine some features of both stocks and bonds and may also include some built-in options. Nevertheless, in this chapter we generally limit our analysis to basic securities: common stocks and bonds.

New shareholders or bondholders have to pay a market price for the newly issued securities offered by companies, so from a company's perspective it is a cash injection equal to the market value of new securities issued (net of any transaction costs). It may seem obvious but is not–companies at early stages may offer securities to their founders or other supporting parties at a price much lower than their market values (and it must be mentioned that estimating the intrinsic value of a startup's stock is always difficult).

Debt securities are also offered mainly to raise funds for future investments but many companies issue new bonds just **to repay current debts**. Replacing maturing debt with a new one is a way of keeping debt to equity ratio at a relatively stable level, which is practiced by many companies. Sometimes companies issue new debt to replace the existing one before the latter matures. It happens mainly when interest rates decline and companies may thus benefit from replacing old "expensive" debt with a "cheaper" one. It still does not affect the ratio of equity to debt.

However, some companies issue debt instruments **to restructure their financing mix**. Companies may decide to use the proceeds from a bond issue **to buy back** and redeem part of their **shares**. It should not affect company's investment policy (at least directly) but it will change the capital structure, the financial leverage and thus the risk borne by all stakeholders. After such a change shareholders can count on higher returns on (lowered) invested capital but they risk more by retaining their stake in now more indebted company. Benefits of financial leverage may be thus offset by a higher risk and thus a higher required rate of return (higher cost of equity).

To what extent (if any) changes in capital structure affect firm value and shareholders' wealth is a crucial issue in the modern corporate finance. Academics present different arguments for or against the hypothesis that changes in capital structure may have impact on firm value by referring to the so-called market imperfections (taxes, information asymmetry, agency costs). Nevertheless, this topic goes far beyond the scope of this chapter–devoted to securities offerings–and thus will not be considered here.

Hybrid securities such as **preferred stocks** or **convertible bonds** also provide companies with funds but give their holders **special rights**. For example U.S. companies that issue preferred stock have an obligation to pay dividends, so preferred stockholders get dividends even when companies generate losses and pay nothing to common shareholders. On the other hand, it typically comes at a cost of no voting right. Convertible bonds give their holders an **option to** decide whether to **convert them into stocks** and become common stockholders or let them be repaid by issuers. Such extra rights or options may encourage outside investors to provide companies with capital. Moreover, by issuing convertible bonds current shareholders retain control over the company and postpone the moment they will have to start sharing voting rights with new stakeholders.

Some offerings include not only newly issued shares (**primary tranche** or **primary offering**) but existing shares held by current shareholders (**secondary tranche** or **secondary offering)**. The aim of such offering is to enable current shareholders withdraw their stake (at least partially) and realize returns–it simply helps existing shareholders **to cash out** and **diversify** their portfolios. Notice that proceeds from such offerings flow directly to leaving shareholders, not to the company. Such offerings are common in the initial (first) public offering–when private companies become public companies. A flotation stimulates significant changes in ownership structure–founders can leave the company but typically they decide to reduce their stake, still retaining the control over the company (by keeping the majority stake).

Raising capital for future investments is the most important aim of security offerings but under some circumstances companies can issue and offer new stocks for other reasons:

- **to award managers:** when stocks are used as a part of compensation packages,

– to pay in shares for stakes in another company in mergers and acquisitions (M&A) transactions (**stock-for-stock deals**),
– **to avoid a hostile takeover:** when newly issued stocks are offered to current shareholders to discourage the acquiring firm (often these newly issued stocks give more voting rights that standard common stocks, however, it is not possible in every country).

4.3. Side effects of stock offerings: Dilution and wealth transfers

Raising capital is the main aim of security offerings. When this capital comes from outside investors such offerings have some side effects for existing shareholders. Selling new shares of the same type (e.g. common shares) to new shareholders will always reduce current shareholders' stake in a company and thus **reduce their voting power**. This effect is called **dilution** of control. Some shareholders may thus prefer debt capital to newly issued shares as a way of raising new funds without affecting their voting power. Dilution of control may be calculated as a **percentage of voting rights transferred to new shareholders**:

$$DoC = \frac{Number\ of\ new\ stocks \times number\ of\ votes\ per\ one\ new\ stock}{Total\ number\ of\ votes\ after\ new\ issue}.$$

Dilution changes the structure of control rights but generally does not affect shareholders' wealth if new shares are offered at a price that reflects their intrinsic value. Selling new securities to new shareholders at a price lower than the intrinsic value would diminish current shareholders' wealth–it would be a ***wealth transfer*** from existing shareholders to the new ones. On the other hand, selling stocks for a price higher than the intrinsic value would cause a similar wealth transfer but in the opposite direction – from the new shareholders to the old ones.

Sometimes another consequence of a new stock issue is pointed out. Even if the newly issued stocks are offered at a current market price, the increasing number of shares will affect some financial ratios (such as **EPS** or **P/E**), especially when they are calculated on historical book data–this effect is called ***accounting dilution***. Some investors allegedly perceive these changes in commonly used ratios as signals of worsening of company's performance, which is simply not true. It is the aim of raising new capital, not the transaction itself, that is crucial for company's future performance. If the raised capital is effectively invested into new projects, contributing to the increase in future profits and free cash flow, it will push the future stock value and thus the market price up. If not, the dilution effect will cause a decrease in dividends per share which can lead to a lowered stock price. So, the issuing *per se* should not affect total firm value or stockholder's wealth. At least when all stakeholders have the same information about companies' future perspectives. What if not? New issues of stocks or bonds may be treated by less informed investors as a bad (stocks) or good (bonds) signal about company's future performance perceived by insiders. This idea refers to one of capital structure theories based on information asymmetry and goes beyond the scope of this chapter.

Example 2

The table presented below shows the above mentioned effects of a new stock issue: reduced voting power of old shareholders, accounting dilution and possible wealth transfer between new and old shareholders on the example of Dollar Inc. Different consequences of newly issued stocks are presented in separate columns: column [3] shows the situation before the issue; column [4] shows the situation after the issue of 0.5m new stocks offered to new shareholders at $30 (intrinsic value); column [4] shows the situation after the issue of 0.5m new stocks offered to new shareholders at $15 (below intrinsic value).

Table 2. Consequences of new stock issue on shareholders' position and financial ratios

	[Units]	Before the issue	After the issue (S1)	After the issue (S2)
[1]	**[2]**	**[3]**	**[4]**	**[5]**
PANEL A				
Number of newly issued stock	[m]		0.5	0.5
Total value of operating assets	[$m]	40.0	40.0	40.0
Value of non-operating assets	[$m]	0.0	15.0	7.5
Total firm value	[$m]	40.0	55.0	47.5
Debt value	[$m]	10.0	10.0	10.0
Market value of equity	[$m]	30.0	45.0	37.5
Number of shares outstanding	[m]	1.0	1.5	1.5
Intrinsic value of one stock	[$]	30.0	30.0	25.0
Offering price for one stock	[$]		**30.0**	**15.0**
PANEL B				
Par value of single stock	[$]	1.0	1.0	1.0
Book value of paid-in capital	[$m]	1.0	1.5	1.5
Book value of retained earnings	[$m]	11.0	11.0	11.0

Table 2 – cont.

	[Units]	Before the issue	After the issue (S1)	After the issue (S2)
Book value of additional paid-in capital	[$m]		14.5	7.0
Book value of total common equity	[$m]	12.0	27.0	19.5
Book value per share	[$]	12.0	18.0	13.0
Net profit for last 12 months	[$m]	1.5	1.5	1.5
PANEL C				
EPS		**1.5**	**1.0**	**1.0**
MV/BV		**2.5**	**1.7**	**1.9**
P/E		**20.0**	**30.0**	**25.0**
Old shareholders' stake (%)		**100.0**	**66.7**	**66.7**
New shareholders' stake (%)			**33.3**	**33.3**
Value of old shareholders' stake	[$m]	30.0	30.0	25.0
Value of new shareholders' stake	[$m]		15.0	12.5
Wealth transfer	**[$m]**		0.0	–5.0

Panel A shows the results of a discounted cash flow (DCF) valuation of company's assets, its equity and its single stock. According to a DCF valuation firm value is the sum of the value of operating assets (assets that generate free cash flow–FCF) and non-operating assets (such as excess cash and marketable securities, unused property etc.). The value of operating assets is estimated as the present value of company's future FCF discounted at weighted average cost of capital (WACC). It is assumed that Dollar Inc. has operating assets worth $40m and no non-operating assets. Its interest-bearing debt accounts for $10m. The value of equity of $30m is a difference between total firm value and the value of financial debt. The company has 1 million shares outstanding so the intrinsic value of single stock is $30 ($30m of equity divided by 1m shares outstanding).

After the new stock issue the total firm value grows by the amount of the new cash raised (for simplicity, we ignore any transaction costs). The cash is treated as a non-operating asset to be used by the company in the future.

In the first scenario (*S*1) we assume that the company issues 0.5m new shares and offers them to new shareholders for a price reflecting their intrinsic (market) value: $30. Proceeds from the new issue ($15m) increase the total firm value to $55m and the value of equity to $45m. The stock's intrinsic value remains unchanged ($45m / 1.5m = $30).

In the second scenario (*S*2) new shareholders pay a price that is lower than the "true" value of the stocks they purchase: $15 per share. In such a case the proceeds of $7.5m would increase the firm value only to $47.5m and the value of equity to $37.5m. Notice that it should eventually decrease the market value of one stock, calculated as the quotient of market value of equity and (increased) number of shares. In our example it drops to $25 ($37.5m / 1.5m).

Panel B simply informs about the accounting treatment of of newly issued shares. Proceeds from the issue will increase the book value of equity by $15m (0.5m stocks issued at $30) in the first scenario and by $7.5m (0.5m stocks issued at $15) in the second one (from assumed $12m before the issue).

Panel C shows the consequences of the new issue from the perspective of current (old) and new shareholders. Dilution leads to a decrease in earnings per share (EPS) from 1.5 to 1.0 regardless of the scenario, because the same amount of (historical) earnings is now divided by higher number of shares. Market to book value ratio (MV/BV), calculated as the quotient of market value of equity to its book value, also decreases, but not to the same extent in both scenarios. Price to earnings ratio (stock's market value divided by earnings per share) rises from 20 to 30 in the first scenario and 25 in the second one, so outside investors may find the stock relatively less attractive if they look only at numbers, ignoring the cause of the change.

After the new issue the stake held by old shareholders drops to 66.6%, which is an obvious result of issuing new shares and offering them to new shareholders. Notice that the same effect is observed regardless of the price paid by new shareholders (we assume that the newly issued shares have the same voting parity as the outstanding shares).

Finally, the last row of the table shows that there is no wealth transfer when stocks are "fairly" priced – the value of the total stake of old shareholders remains unchanged (*S*1). However, in the second scenario (*S*2) we can see a decline in the value of old shareholders' stake from $30m before the offering to $25m after it (1 million shares, each worth $25). What we see is a wealth transfer from the old shareholders to the new ones who now own a stake worth $12.5m (0.5 million shares, each worth $25) for which they have actually paid $7.5m.

To avoid possible wealth transfers and to prevent the weakening of old shareholders' voting power, managers–who are expected to act in the best interest of current shareholders–often decide to offer newly issued securities only to existing shareholders in order to raise more capital without affecting ownership structure. Moreover, in many countries current shareholders have priority right to newly issued stocks guaranteed by law–the so called **preemptive rights**. We will refer to this later on when we introduce the so called ***rights offerings***.

What if existing shareholders do not have capital or do not want to employ capital to the company but are still interested in retaining control? They may offer new shares to new shareholders but limiting their voting power. This can be done by issuing different **classes of shares**. In some countries companies may issue shares with lower/higher voting rights (or no voting rights at all) and offer them to new shareholders. If a company issues new shares with lower parity (e.g. 1/10) and offers them to new shareholders, it is able to raise required capital with current shareholders retaining their control over the company. The question is why new investors would accept such an offer. The answer is quite simple–**many shareholders of public companies never exercise their voting rights** but always exercise their cash flow rights (rights to receive dividends). So it seems reasonable to give such shareholders exactly what they want, especially when current shareholders care deeply about retaining control.

4.4. Security offerings and companies' life cycle

Both the aim and the type of a security offering depend heavily on firms' life cycle and their status (whether or not they are listed). Young, private firms are financed solely with funds provided by their founders or founders' families. These firms issue securities almost exclusively to raise capital and their offerings are private, which means that newly issued securities are offered to a small group of investors or even to a single investor that specializes in providing funds for private companies. It may be a **business angel** (a wealthy individual interested in providing the so-called **seed capital** for startups), **venture capital firm** (VC) or **private equity** firm (PE). However, during the last decade another way of raising capital gained popularity among startups: **crowdfunding**. Basically, crowdfunding may not even include any security offering, because providers of funds (called backers rather than investors) can be rewarded in many different ways (e.g. obtaining discounts when buying company's products).

At a later stage a company that wants to raise more capital to finance its growth has a few more options. Generally, it can issue and offer its securities (stocks, bonds and hybrid securities) in three ways depending on the final target:

- in the form of a **private placement**, when securities are offered only to a small specified group of knowledgeable investors,
- in the form of a **rights offering**, when securities are offered only to existing shareholders,
- in the form of a **public offering**, when securities are offered to general public on the open market via stock exchange (it is also called **general cash offer**).

When a private company decides to offer securities publicly, it becomes a public company, whose securities are traded on the open market. Public companies have much better access to capital and much more options of raising it. Moreover, as we pointed out in the previous subchapter, public companies may offer securities not only to raise additional capital but for many different reasons. They still can use all three methods: rights offering (when they offer new shares to existing shareholders, e.g. to avoid a hostile takeover); private placements (especially when they issue bonds to a small group of financial institutions) and subsequent public offerings (when they offer newly issued shares on the open market to raise capital).

Public offerings are more complex and more time-consuming than private placements. Newly issued stocks or bonds that are **publicly offered have to be registered** by a given regulatory authority, such as Securities and Exchange Commission (SEC) in the U.S. or Polish Financial Supervision Authority (KNF) in Poland. Not all the companies meet the requirements set by regulators to register their securities, so this type of offering may be not available to many companies.

In public companies the **decision about new security issue** and its form is made **by managers** (executives) but in some legal systems it has to be approved by existing shareholders. Generally, in the U.S. managers (or being precise–a board of directors) have far-reaching flexibility in issuing new securities. Moreover, the power of deciding to whom the newly issued shares are offered and what type of offering is used is practically in the hands of the board of directors, mostly because of **limitations of shareholders' preemptive rights**. In Europe, on the contrary, **shareholders** must **approve managers' propositions**. Their preemptive rights are generally given as a general principle (the so-called default rule). Moreover, in some countries any increase in stock capital has to be approved by shareholders' meeting with **supermajority**, so managers' discretion is thus curbed to a significant extent.

4.5. Types of offerings

4.5.1. Private placements

As we mentioned before, capital market regulatory and supervisory authorities register all publicly offered securities. The registration procedure may be quite complex, costly and time-consuming. This does not refer to private placements, which is the biggest advantage of this method of security offering. Offerings are treated as private placements when they target a **specified and limited group of investors** Currently (2018) the maximum acceptable number of targeted investors equals **35** in the U.S. and **149** in Poland. An offer to an undefined group of investors or an offer to a defined group of investors exceeding a certain number (e.g. to 150 or more investors in Poland) is treated as a public offering.

Private placements are generally much **easier to arrange** and require **much less information** to be revealed by the issuing company. They are also less time-consuming than public offerings, which means that additional **capital can be raised relatively quickly**. Moreover, companies' managers (in private companies simply their owners) may decide to whom new securities are offered so they can "set" the ownership structure (to some extent it is also possible in public offerings but with limitations).

The main disadvantage of private placements is a **limited secondary market**. Any investor deciding to buy bonds or stocks in a private placement should be aware that it may be quite difficult to exit, because it is not so easy to find a buyer outside the open market. It is a problem for investors but also for issuing companies – to attract investors they have to compensate them for this lack of liquidity by offering **extra premium** to the rate of return. That's why it is not obvious whether private placements are "effectively" cheaper than public offerings. On the one hand, direct costs of organizing an offer (fees for legal advisors, auditors and underwriters, commissions paid to authorities and stock exchanges, etc.) are lower but, on the other hand, cost of capital (e.g. an offered coupon rate in bond offering) is higher.

Private placements are **used mostly for debt instruments**, especially in the U.S., where bonds are offered to institutional investors, like pension funds, mutual funds or insurance companies. Notice that bonds have strictly defined maturity, so these investors will be eventually paid off when the bonds mature without a need to resell them (of course, they will have to find a buyer, if they want to exit before maturity date). It does not refer to stocks which have undefined maturity, so to exit, one needs to find a willing buyer.

Since the '90s this method has been also used extensively by mature European companies and companies from other parts of the world to issue both stocks and bonds in the U.S. due to some relaxed restrictions in subsequent trading of such securities. In 1990 SEC adopted **Rule 144A** under which securities can be offered in the form of a private placement to U.S. big financial institutions (with at least $100m in assets under management), called **qualified institutional buyers** (**QIBs**), who can trade unregistered securities among themselves before they are eventually traded on the open market. Relaxing the trading restrictions significantly broadened the US private equity market (and bond market, especially for non-U.S. companies).

Polish listed companies use private placements mainly to raise additional capital in a special form called target capital (unknown in common law systems). Shareholders can delegate to managers the power to increase the initial share capital by the amount that cannot exceed 75% of the outstanding shares (similar rules exist in other European countries, including Germany, Italy and many more but they can differ slightly). Managers can exercise this right and increase the share capital by issuing and offering new shares to old or new shareholders. In practice, they typically start with offering subscription warrants to a specified group of investors. These warrants can be eventually converted into company's shares. The procedure allows to raise capital relatively quickly with the transaction costs limited to the minimum, which is its main advantage, as pointed out by managers of public companies listed on the Warsaw Stock Exchange. It gives managers valuable financial flexibility, but at the cost of limited shareholders' rights. Minority investors often claim that the procedure leads to the dilution effect because it allows to exclude their preemptive rights (this is one of the exceptions from the general principle when current shareholders cannot exercise their preemptive rights to new shares).

4.5.2. Initial public offerings

4.5.2.1. Basics of IPOs and reasons for going public

Initial public offering (IPO) is a company's first security offering to the public (unknown investors). It may refer to both stocks and bonds, however, typically one means stock offering when talking about IPO so we will also do this throughout this chapter.

After the IPO a private (closed) company becomes a **public company** (also called a **listed company**), because from that time on its shares (bonds) are traded on an open market and thus are easily available to the public. **Going public** (called also simply **debut** or **flotation**) is typically a milestone in company's life cycle.

First of all, it causes a big change in company's **ownership structure**. After the IPO the number of company's owners may increase from several founders to thousands or even millions of stockholders. It obviously makes the ownership structure of public companies much more dispersed, meaning that an average stockholder holds a relatively small stake in company's capital. Old shareholders lose some of their voting power but typically retain control over the company even when they sell part of their stake during IPO. Moreover, it is not only the number of shareholders that grows after IPO, but also the frequency the stakes trade between subsequent stockholders. The ownership structure of public companies is changing all the time as millions of its shares can be traded daily on the open market.

Secondly, a public company needs to change totally its attitude towards disclosures. Closely held companies typically keep all the information about their activities secret, unless some information is required by special rules (it may refer to financial statements informing about companies' financial performance and financial position). Public companies, on the other hand, have to meet

mandatory disclosure and reporting standards, meaning that they have to reveal all important information (information that could affect its market price) to the public. It is required by regulatory authorities to give all the outside investors the same opportunity to use the information in their investment decisions. Actually, the reporting requirements for public companies imposed by regulatory authorities are the main reason why the owners of many big, mature companies still prefer their firms to operate as closely-held companies.

We started this chapter with indicating the main aims of security offerings highlighting the company's need for additional capital. In the section devoted to IPOs we should add some special aims of this special form of offering. Why do companies go public? Of course, most of them do this because listed companies have better access to capital. It is why relatively young firms with substantial capital need to go public, especially in the U.S. where the average age of such companies is much lower than in Europe.

Nevertheless, it is definitely not the only reason why companies organize IPOs. First of all, main shareholders often use an IPO as the opportunity to **cash out**, an important reason which we also mentioned earlier. It refers especially to venture capital and private equity firms (**VC/PE** firms) that invest in young private companies, sometimes at very early stage (e.g. startup) and exit often via IPOs. Other group of investors that often cash out during IPOs are companies' founders. However, founders typically sell a relatively small part of their stakes, contrary to VC/PE firms that basically get rid of their whole stake in companies going public.

In practice

Dino Polska SA is a Polish retailer that went public in 2017. It was one of the biggest IPOs in the last several years. A secondary tranche (the only one) consisted of shares held (indirectly) by one of the biggest private equity firm that operates in Poland – Enterprise Investors. They were sold for PLN 33.50, which gives an offer the size of PLN 1.6bn. Enterprise Investors had invested about PLN 200m seven years before the company went public. It gives a holding period pre-tax rate of return of app. 700% (no dividends were paid during that period).

IPOs are sometimes used also to establish market price of stock and to let the market asses company's performance. This reason is highlighted when companies engage in **equity carve-out** transactions. Equity carve-out is a form of a divestiture when a company sells to the public its stake in a subsidiary, typically a business segment previously separated from the parent company that operates in different industry.

Last, but not least, managers often claim that one of the reasons for organizing an initial public offering and going public is to gain publicity and enhance company's reputation. It is not easy to asses if there is any "marketing" gain from going public, but it seems reasonable that it can enhance companies' credibility, mainly due to the transparency forced by reporting standards.

4.5.2.2. IPO procedures and requirements

Decision to go public. The very first step in IPO is always the decision to go public, which is typically proposed by managers and approved by shareholders (notice that in many private, closely held companies it may be the same individuals). The formal approval may take the form of a General Shareholders' Meeting's resolution. It is worth mentioning that those firms that operate in another form than a corporation (e.g. partnerships) have to **incorporate** before they go public. Not all the companies are eligible to go public. They have to meet the criteria set by stock exchanges, called **listing requirements** covering generally three areas: **financial history, size** and **liquidity**. Many stock exchanges require companies to operate for at least 3-5 years before going public. Most exchanges also require minimum earnings level or **minimum market capitalization** (total number of shares outstanding multiplied by the share market price). Companies have to submit financial statements documenting their financial performance to regulatory authorities and to stock exchanges for approval. Stock exchanges care about future trading so they require minimum liquidity guaranteed by the company. It is typically measured with the so-called **free float** which is the proportion of outstanding stocks available for trading after all blockholders' stakes are excluded.

In practice

*In many countries younger and smaller companies that cannot be listed in the **main markets** may still choose **sub-markets** with less strict requirements. The best examples are **AIM** (Alternative Investment Market) run by the London Stock Exchange with 960 companies listed at the end of 2017 and total capitalization exceeding GBP 100bn or **New Connect** run by the Warsaw Stock Exchange with 400 companies and total capitalization of approximately PLN 10bn. It is worth mentioning that some companies listed in sub-markets implement relatively poor corporate governance structures and even stop meeting the limited listing requirements (e.g. stop publishing financial reports) and are eventually excluded from listings. Sub-markets are generally full of the so-called **penny stocks**. Deutsche Boerse was forced to close its sub-market, **Neuer Markt**, in 2002 to improve market transparency and regain investor confidence.*

Choosing underwriter(s) and other advisors. After the decision is made, the company should choose advisors and auditors. They are responsible for **business** and **legal due diligence** of the company. The most important role is played by an **investment firm**, typically an **investment bank** (or a group of banks) that arranges the whole offering. It is responsible mainly for marketing and selling securities offered in IPO, but it may be also responsible for stock valuation, setting the final offering price and allocating shares between investors. It may also **underwrite** the securities taking the risk of low demand from investors, but this service is offered only when offering is made on the so-called **firm commitment** basis. We call the investment banks that help companies go public **underwriters**, however, to be precise we should not, because they not always serve as actual underwriters. The scope of investment bank's tasks and responsibilities depends on the basis on which the cooperation between an issuing firm and an underwriter is set. It will be explained in detail in the next section.

Prospectus. Companies that go public are generally unknown to outside investors so regulators require detailed information on their business profile, management, past performance, resources they use and risks they face. Such details are included in a **prospectus**, a formal document that has to be enclosed during the registration procedure and revealed to the public. The main aim of a prospectus is to give the outside investors an overview of a company and an offering itself. A **preliminary prospectus** (also called a **red herring prospectus**) includes the above mentioned details but also the structure and **aim of the offering**. In some countries a prospectus should also include company's future plans and financial forecasts. Company's **financial statements** for the last several years are enclosed to the document. At a later stage, after the registration is approved and the final offering price is set, a **final prospectus** is revealed with complete information about all the details of IPO, including the **number of shares eventually offered** in each tranche (if there are separate tranches for different groups of investors), **their price** and the **rules of allocation**.

Registration statement. Registration procedure requires companies to file a **registration statement** with a regulatory authority. The scope and details of a registration statement are established by regulators. Generally, it includes the **preliminary prospectus** and **other documents** with extra details that do not have to be revealed to outside investors. The regulatory authority studies the registration documentations during the so-called **waiting period**. In the meantime the company begins marketing activities by revealing the preliminary prospectus and goes through other formal requirements imposed on listed companies (agreements with stock exchange, depository institution, etc.). The registered securities are being given the identification number–**ISIN** (International Securities Identification Number). Apart from 12-character alpha-numerical ISIN, publicly traded securities have shorter (typically three- or four-character) unique, local **ticker symbol** assigned by a stock exchange.

Road show, book-building and setting the final offering price. To attract institutional investors a series of meetings with potential investors is arranged during which company managers present the offer. This is also an opportunity for underwriters to make first estimates of investors' demand for offered shares. During the road show underwriters collect **non-binding bids** in the form of an **order book** from outside investors who reveal the information about the volume and the price they are willing to pay for the shares within the **preliminary price range** (of approximately 10-15%) set by underwriters and approved by issuers. This procedure is called book-building and is used **to help** underwriters **set the final price** for the shares and allocate the shares between institutional investors. After the order book is closed, the book-runner analyzes the information about possible demand revealed by participating investors. Then the final price is set and shares may be distributed to investors if they accept it.

It is worth mentioning that book-building is not the only way of setting the final stock price, however it is now the most popular method all over the world. There may be **fixed-price offers** with a fixed price set without a book-building phase. This method, very popular in the 80s, has one important disadvantage. The **risk of unsuccessful offering is relatively higher** than in book-building, because issuers and investment banks have **no feedback** from outside investors before the price is set. The offer price is based mainly on the stock valuation, so this phase's role is crucial in this method. However, it is also extremely important in book-building where stock intrinsic value is used as a benchmark for setting the preliminary price range. Moreover, to avoid this risk issuers together with investment banks may decide to reduce the offering price below its intrinsic value. This discount represents an opportunity cost and leads in many cases to the so-called **IPO undervaluation** (when stocks' prices rapidly go up in the immediate aftermarket). This phenomenon is also observed with offerings based on book-building but with a fixed-offer price it may lead to much more "**money left on the table**".

Another way of pricing shares in IPO is based on auctions, especially the so-called **Dutch auctions**, which are used to set a **single final price** but in a way that differs from book-building. First of all, only a minimum price is set and all investors are asked to place their bids at or above this minimum price. Investors' orders are collected as a set of different volumes of shares priced at different levels starting with the highest price and ending with a minimum price. On the basis of these series of orders the **clearing price** is set, that is **the highest possible price** that **guaranties sufficient proceeds** or guaranties that a certain number of shares will be sold. Eventually, all investors bidding at or above the set price get their shares **for that clearing price**, so a Dutch auction encourages investors to bid high. Moreover, it allocates shares automatically, which is the main difference between auctions and book-building, because the latter enables allocating shares discretionally. Dutch auctions are not so popular, however they may be used even for relatively large offers. The most famous IPO in which this method was used was **Google's IPO in 2004**.

Allocation of shares. The final step before the first day of trading is allocation of shares (also called **allotment**). Generally, shares may be offered in **two tranches**: one for **institutional investors** and one for **retail investors**. However, in some countries only institutional investors are included in the offer. In the most popular book-building method the final price is set on the basis of bids recorded in the order book and for this price shares are eventually offered to institutional investors. When shares are offered partly to retail investors, the price is basically the same, however in some countries retail investors get a 2-3% discount. Retail investors typically subscribe for shares via their brokerage firms setting the number of shares they are willing to buy. During the subscription period they typically know only the preliminary price range, similarly to institutional investors. The final offer price is set after the subscription period is closed (a few days after the book-building phase is closed).

The way the shares are allocated between institutional and retail investors differs significantly. Under book-building offerings underwriters and issuer's managers use **discretion** to **allocate shares between institutional investors**. Of course, shares are allocated between those investors that participated in book-building but issuers typically favour long-term investors, such as pension funds, because of relatively low risk of the so-called **flipping** (sale of stocks in the immediate aftermarket). Under auction offerings the allocation is generally automatic, as we mentioned before. An allocation between retail investors is typically proportional (based on a pro-rata basis). In case of an oversubscription, the number of stocks allocated to a given investor is proportionally reduced.

In practice

When GPW S.A. (the state-controlled operator of the Warsaw Stock Exchange) went public in 2010 its newly offered shares were divided into two tranches: one for institutional investors (70%) and one for retail investors (30%). The offering price was set at PLN46 for institutional investors (the demand was about 25 times higher than the number of shares offered) and at PLN43 for retail investors. GPW's IPO differed from other Polish IPOs with regard to retail investor tranche. These investors were allowed to subscribe for no more than 100 stocks. However, those who did so eventually got only 25 stocks after necessary reductions were made due to oversubscription.

First day of trading. A few days after the allocation of shares the trading on a stock exchange starts, so existing shareholders can sell and new shareholders can buy shares in the secondary market. It is worth mentioning that some shareholders and an underwriter may have certain obligations referring to the trading.

Aftermarket. The short period after the issue is called aftermarket. Underwriters often agree to **stabilize the price** during this period, so they are obliged to buy stocks on the open market in case of strong supply, thereby supporting the price. Generally, they also agree not to sell any shares below their offering price in the aftermarket. Far-reaching limitations refer also to main shareholders (e.g. company's founders). They usually accept the so-called **lock-up provision** prohibiting selling shares before a specified deadline (e.g. six or twelve months). This provision is also included in the prospectus. All these obligations aim at reducing the risk of a price fall at least in the short- and medium-term. Once the lock-up period ends insiders are allowed to sell their shares, however they have to reveal such transactions to the public regardless of the number of shares sold. These information revelation is required by supervising authorities to enable less informed shareholders make sound investment decisions based.

In practice

JSW SA–a Polish mining company went public in 2011. The offering price was set at PLN136. The price fell in the aftermarket. During the first 30 days of trading two investment firms (owned by banks): DM Bank Handlowy and DM PKO BP were trying to stabilize the price buying shares at PLN135.3-136. Two months after the first day of trading the shares traded at a price below PLN100 and three years later below PLN50.

4.5.2.3. The role of investment banks in IPO

Investment banks play a key role in IPO, designing its structure and arranging the whole process. Typically several banks are involved in one offering, especially when offering's size is relatively big. They form a **syndicate** with a **lead manager** coordinating the whole offering and other banks playing less important roles, sometimes limited to selling securities to their clients and underwriting certain portions of securities offered. The main areas for which investment banks take the full or at least partial responsibility in IPO include among others:

- pursuing company due diligence,
- preparing stock valuation,
- attracting investors (marketing),
- setting the final stock price,
- selling the offer,
- underwriting securities,
- allocating shares,
- stabilizing the stock price in the aftermarket.

The most important stages are: selling the offered securities to outside investors and underwriting these securities. Generally, there are two kinds of agreements between issuers and investment banks that define the scope of responsibility of investment banks and risks born by both parties: **firm commitment** and **best efforts**.

On a **firm commitment basis** an investment bank (or a syndicate) simply buys the offered shares from the issuer and then sells them to the public, taking the risk of low investors' demand for shares. The bank thus **underwrites** the securities offered, so it serves as an **underwriter**. Underwriters earn the so-called **spread** (or discount) between the offering price paid by final investors and the lowered underwriter's buying price. Their compensation may also include other fees, if they participate actively in other steps of IPO process, which refers mainly to the lead manager. As we pointed our earlier, underwriters **take the risk** that outside investors are unwilling to buy shares at the offering price. However, underwriters typically benefit in the opposite situation, when demand for stocks is strong, which pushes the prices up. It is because they often have an option to buy additional shares from the issuer at the offering price (typically up to 15% of the primary offer). This option–called **Green Shoe provision** (after the name of the first company that used it in its IPO) or **overallotment provision**–is exercised when stock price goes up in the aftermarket (the option is **in-the-money**) but underwriters have limited time to use it (typically one month).

Under a **best-efforts offering** an investment bank does not guarantee that all the stocks will be sold, instead it **promises** to do its best **to sell as many stocks as possible**. If a best-efforts offering is used, issuers set a minimum required number of stocks to be sold to make the offer valid (e.g. when less than 40% of the offered stocks find their buyers, the whole offer may be cancelled). The bank's compensation is set as a fee for a given amount of stocks sold.

In practice

Spotify went public in 2018 but without any support of investment banks. It was the so-called ***direct listing*** *(unusual for companies of this size), so the existing shares just started trading on NYSE without any price-setting procedure and without hefty fees for investment banks.*

4.5.3. Secondary (seasoned) public offerings

After an IPO companies become public and thus have easier access to capital. To raise additional capital they may organize another public offering and sell newly issued securities to new shareholders. Such offering is called **secondary public offering** (SPO) when it refers to both stocks and bonds, or **seasoned equity offering** (SEO) when it refers to stocks only.

The procedure is very similar to IPO and also includes a new prospectus and registration statement but there is one big difference between SEO and IPO–the current market price is set on the open market so there is no need for a complex price-setting procedure. Similarly to an IPO, the secondary public offering may include newly issued stocks (primary tranche) or existing stocks (secondary tranche). Of course, only with the primary tranche the company raises capital and gets proceeds that can be further invested. The secondary tranche is used when some main shareholders want to withdraw their stakes. Costs of arranging SEOs are on average lower than for IPOs.

There are two interesting aspects worth mentioning with regard to SEOs. The **stock market reaction** to companies' announcements of new shares to be issued to new shareholders is typically **negative**, which means that stock prices typically go down after such announcements. Moreover, companies that organize SEOs tend to **underperform in the long-run**. According to capital structure theory the reasons for that may stem from information asymmetry. If the so-called **insiders** know more about company's future perspectives that outside investors, the investors may treat the **new equity issue as a bad signal**, contrary to a stock buyback treated as a good signal. Why? Outside investors may expect that the current price is higher than the stock intrinsic value (stock is overpriced) and managers try to raise capital at a relatively low cost by issuing new stocks at the current market price. The effect is similar to the effect of a market reaction to the information about a stock sell made in the secondary market by an insider (e.g. CEO). Such transaction may also indicate that shares are overpriced and managers know more about company's future poor perspectives. This problem is called **adverse selection.**

4.5.4. Rights offerings (issues)

With rights offering, newly issued shares are offered to existing shareholders proportionally to their current stakes in stock capital. In most European countries all shareholders have **preemptive rights** given by law, so companies cannot generally organize a public offer excluding their current shareholders–newly issued shares have to be offered to the existing shareholders first to **protect them from a dilution of control** (mentioned at the beginning of the chapter). However, there are some exceptions to this general principle. Managers are allowed–under certain conditions–to organize a private placement excluding all or some part of the current shareholders. Such exception refers, for example, to stock issue in stock-for-stock mergers when acquiring company issues new stocks to pay with these stocks for the stocks of the target company. In **the U.S.**, on the contrary, the situation is totally different. Managers have more flexibility when deciding about new issues and offerings, because current shareholders have preemptive rights only if they are included in company's **articles of incorporation**. In fact, American public companies rarely use rights offerings – newly issued shares are usually offered publicly. In contrast, European companies use basically rights offerings when they issue new shares to raise additional capital.

Under rights offering existing shareholders are given the option to **buy newly issued shares** on a pro rata basis paying a **subscription price**, which is set below the current market price. If they decide to subscribe, they will be given subscription rights, usually on a one-right-one-stock basis (one right per one stock held). For subscription rights are **separate financial instruments** that can be bought and sold on an open market, the subscribing shareholders may decide to hold rights and use them to buy new stocks or to sell them on an open market. They may, of course, ignore subscription and do nothing. Interestingly, if they subscribe, they can sell the rights or keep them to exchange for shares paying subscription price–regardless of the strategy, they will probably be neither better nor worse off. If they choose not to participate in rights issue, their wealth will be reduced. We will illustrate this point with a simple example.

Example 3

Euro Inc. has one million stocks outstanding priced currently on the open market at $50, which gives $50m of market value of equity. **Panel A** of Table 3 presents the data.

The company wants to raise additional capital up to 20% of equity current market value ($10m). Managers decided to offer rights to current shareholders. One stock will entitle a shareholder to get one right, so the number of rights to be issued is equal to the current number of shares outstanding:

Number of rights = Parity · Number of stocks outstanding,
Number of rights = 1 · 1m = 1m.

The subscription price of a new stock is set at $40, which means that to raise $10m of new capital the company has to issue 0.25m new stocks:

Number of new stocks = Expected proceeds from the issue / subscription price,
Number of new stocks = $10m / $40 = 0.25m.

Number of rights needed to buy one new stock is calculated as a quotient of the number of rights to be issued to the number of new stocks:

Number of rights needed to buy one new stock = Number of rights / Number of new stocks,
Number of rights needed to buy one new stock = 1m / 0.25m = 4.

Panel B of Table 3 presents the results of the above calculations.

To calculate the market price of a single subscription right, one should first calculate the expected fall in stock market price after the issue (it should be obvious that when new stocks are offered with a discount, the current market price of stocks should fall after the issue). To calculate the future stock price, we have to divide the market value of equity after the new issue ($60m = $50m + $10m) by the total number of shares after the issue (1.25m = 1m + 0.25m):

Current market price after the issue = Market value of equity / total number of shares,
Current market price after the issue = $60m / 1.25m = $48.

In our example one should expect a fall by $2 from $50 to $48. This fall in price is the value of a subscription right:

Value of subscription rights = Stock price before the issue – Stock price after the issue,
Value of subscription rights = $50 – $48 = $2.

Table 3. Consequences of subscription rights issue

PANEL A	[Units]	Before the new stock issue	
Number of shares outstanding	[m]	1.0	
Market price of a single stock	[$]	50.0	
Total market value of equity	[$m]	50.0	
PANEL B	[Units]	Details of rights offering	
Capital to be raised by the issue (20%)	[$m]	10.0	
Subscription price of a new stock	[$]	40.0	
Number of newly issued stocks to raise planned capital	[m]	0.25	
Number of subscription rights per one share		1	
Number of rights to be issued	[m]	1	
Number of rights needed to buy one new stock		4	
Value of a subscription right	**[$]**	**2**	
PANEL C	[Units]	After the new stock issue	
Number of shares outstanding	[m]	1.25	
Market price of a single stock	[$]	48.0	
Total market value of equity	[$m]	60.0	
PANEL D	[Units]	Investor X	Investor Y
Number of shares before the issue		100	100
Number of subscription rights		100	100

Table 3 – cont.

PANEL D	[Units]	Investor X	Investor Y
Number of new shares		25	0
Total number of shares after the issue		125	100
Total value of shares after the issue	[$]	6 000	4 800
Less the price paid for extra shares	[$]	–1 000	0
Plus the price obtained from selling rights	[$]	0	200
Total position of investor after the issue	**[$]**	**5 000**	**5 000**

Notice, that the value of a single subscription right may be calculated alternatively as:

Value of subscription rights = (Stock price after the issue – Subscription price) / Number of rights needed to buy new share,
Value of subscription rights = ($48 – $40) / 4 = $2,
or
Value of subscription rights = (Stock price before the issue – Subscription price) / (Number of rights needed to buy new share + 1),
Value of subscription rights = ($50 – $40) / (4 + 1) = $2.

Panel D shows the positions of two investors holding initially the same number of stocks (100).

Investor X decided to participate in rights offering and uses these rights to buy new shares whereas investor Y, who also participated in rights offering, decided to sell them on the open market.

Investor X gets 100 subscription rights and then submits the rights in exchange for new shares (25 = 100/4) paying a subscription price of $40 per share. She ends up holding 125 shares worth $6,000 (125 · $48). After subtracting $1,000 paid for additional 25 shares it gives $5,000.

Investor *Y* also gets 100 subscription rights and then sells them on the open market for $2 per one subscription right obtaining $200. She ends up holding still 100 shares worth $4,800. After adding $200 received from selling of rights it also gives **$5,000.**

Notice that if the investors decided not to participate in rights offering, they would end up holding still 100 shares worth $48 each, so they would be worse off.

As we showed in our example, rights offerings always lead to a stock price fall when subscription price is lower than the current market price, which is always true, because if it was higher, no one would be interested in purchasing new shares via subscription rights. Companies never set the subscription price equal to the current market price because even during the relatively short period of time, when shareholders can decide whether to participate, the market price may change, falling below the subscription price, making the whole offer pointless.

This anticipated stock price fall is included in stock quotations. Stock exchanges reduce the stock reference price on the so-called **ex-rights date** by the value of a subscription right, similarly to the reductions made on the ex-dividend date by the value of dividend per share. Of course, it is only a technical change of **previous day's close price** – actual price on ex-rights date may go up or down reflecting instantly changing market conditions.

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CHAPTER 5
MODELING A TERM STRUCTURE OF INTEREST RATES

5.1. Interest rates and bonds

5.1.1. Compounding and discounting

The value of a monetary unit changes with time: one Polish zloty (PLN) today is worth more than the same amount of money next year. Usually it is assumed that people prefer to have money now than in the future. This idea is called "time value of money". To compare cash flows at different dates we should be able to calculate future values of current cash flows (compounding) as well as current values of cash amounts that will be paid in the future (discounting). In this sub-chapter we recall basic formulas for compounding and discounting.

In **simple compounding** interest is the product of the length of a time period (in years) and the interest rate. Assuming that $F(t,T)$ is the interest rate (annual) for the period starting at the moment t (now) and ending at the time T, the length of investment period is $T-t$ years. Placing $K(t)$ with the given interest rate, after $T-t$ year we will have

$$K(T)=K(t)\big(1+(T-t)F(t,T)\big). \tag{1}$$

The $K(T)$ is the future value of the today's $K(t)$ units of currency. From the formula (1) we can obtain current value of $K(T)$ units of currency at the moment T, given by the following formula for **simple discounting**:

$$K(t)=\frac{K(T)}{1+(T-t)F(t,T)}. \tag{2}$$

In the simple compounding it does not matter how often interest is calculated, because it is not added to the invested capital.

In many investments interest is added to the invested capital. In this case after the end of the first compound period, the amount invested is increased by the interest paid after this period. In the second period interest is calculated for this larger amount and then added to the amount invested, and so on. By $Y_k(t,T)$ we will denote the interest rate for the period starting now (i.e. at the moment t) and ending at the date T with compounding k times a year. The final value of initial investment in amount $K(t)$ is given by the following **compounding formula**:

$$K(T)=K(t)\left(1+\frac{Y_k(t,T)}{k}\right)^{k(T-t)}. \tag{3}$$

The current value at the moment t of $K(T)$ units of currency paid at the moment T is given by the **discounting formula**, which can be simply derived from equation (3):

$$K(t)=\frac{K(T)}{\left(1+\frac{Y_k(t,T)}{k}\right)^{k(T-t)}}=K(T)\left(1+\frac{Y_k(t,T)}{k}\right)^{-k(T-t)}. \tag{4}$$

In many cases interest is compound annually. Annually compound rate we denote by $Y(t,T)=Y_1(t,T)$.

We can consider what happens when the compounding period becomes very small or, equivalently, the number of compounding periods in a year becomes large: interest is calculated in every moment. As k tends to infinity, we obtain the following formula for **continuous compounding**:

$$K(T)=K(t)\exp\big((T-t)R(t,T)\big), \tag{5}$$

where $R(t,T)$ is **continuously compounded interest rate** for the period starting now (i.e. at the moment t) and ending at the date T. The current value of $K(T)$ units of currency at the moment T **continuously discounted** is

$$K(t)=K(T)\exp\big(-(T-t)R(t,T)\big). \tag{6}$$

5.1.2. Discount factors

The current value of $K(T)$ units of currency paid at the moment T can be generally described by the discounting formula:

$$K(t)=K(T)P(t,T). \tag{7}$$

where $P(t,T)$ is called a **discount factor** (from the moment T to the moment t). It is a current value of one unit of currency paid at the moment T. Equivalently, it is a number of units of currency at time t that accrues to a unit amount of currency at time T. Obviously, $P(t,T)$ increases as T decreases towards t and $P(t,t)=1$.

The exact formula for the discount factor depends on the type of discounting. The formulae for simple, compound and continuous discounting factors can be easily obtain from equations (2), (4) and (6), respectively, by taking $K(T)=1$. We can also reverse the formulae and solve them for interest rate. Thus, if we know the value of the discount factor, we can calculate the appropriate interest rate. The formulae are given below. The word "spot" is used to distinguish described interest rates from forward interest rates, which will be defined later.

Different types of (spot) interest rates

Simple (spot) rate:

$$F(t,T)=\frac{1-P(t,T)}{(T-t)P(t,T)}. \tag{8}$$

Compound (spot) rate with k compound periods yearly:

$$Y_k(t,T)=k\left(P(t,T)^{-1/[k(T-t)]}-1\right). \tag{9}$$

Continuous (spot) rate:

$$R(t,T)=-\frac{\ln P(t,T)}{T-t}. \tag{10}$$

Example 1

Assume that the discount factor for the period that ends two years from now equals $P(0,2)=0.81873$. The different interest rates calculated based on this discount factor are as follows:

- Simple rate:

$$F(0,2)=\frac{1-0.81873}{2\cdot 0.81873}=0.1107=11.07\%.$$

- Compound rate (with annual compounding):

$$Y_1(0,2)=0.81873^{-\frac{1}{2}}-1=0.1051=10.51\%.$$

• Compound rate (with semiannual compounding):

$$Y_2(0,2) = 2 \cdot (0.81873^{-\frac{1}{4}} - 1) = 0.1025 = 10.25\%.$$

• Compound rate (with quarterly compounding):

$$Y_4(0,2) = 4 \cdot (0.81873^{-\frac{1}{8}} - 1) = 0.1013 = 10.13\%.$$

• Continuous rate:

$$R(0,2) = -\frac{\ln 0.81873}{2} = 0.1 = 10\%.$$

Let us consider a zero-coupon bond, i.e. a financial instrument that will pay a specified amount of money at the maturity date T. We assume that the amount that will be paid is 1 PLN and that it will be paid for sure: there is no risk of default. The bond is traded on the market. It is easy to see that its current price should be equal to the discount factor $P(t,T)$ as the bond is an equivalent of 1 PLN paid at the maturity T. In theoretical models, it is assumed that for any moment $T > t$ there exists a zero-coupon bond with maturity T, which can be bought and sold on a frictionless market (i.e. there are no transaction costs, the market is perfectly liquid and one can buy and sell any amount of bonds). For any moment t the prices $P(t,T)$ expressed as a function of maturity time form a **term-structure of zero-coupon bond prices** or **discount curve**.

5.1.3. Forward rates

Let us consider three moments of time: t, S and T, where $t < S < T$. At moment t we buy one zero-coupon bond with maturity T, which requires $P(t,T)$ PLN. To finance this purchase we sell the zero-coupon bonds with maturity S. As one bond costs $P(t,S)$ PLN, to obtain $P(t,T)$ PLN we should sell $P(t,T)/P(t,S)$ of such bonds. The total value of these transactions is zero: we do not have to use any additional money and no money is left. But we have obligations connected with the bond that matures at moment S and receivables connected with the bond maturing at T. At moment S we will have to pay $P(t,T)/P(t,S)$ PLN. At moment T we will obtain 1 PLN. Figure 1 presents cash flows connected with the described transactions.

As one can see, this pair of transaction is financially equivalent to investing $P(t,T)/P(t,S)$ PLN at moment S in order to obtain 1 PLN at time T. The discounted value of 1 PLN paid at moment T is $P(t,T)/P(t,S)$ PLN and thus $P(t,T)/P(t,S)$ is the discount factor from moment T to moment S. Using again the equations (2), (4) and (6) we can calculate the interest rates for the period between two moments in the future. This type of interest rates is called **forward rates**.

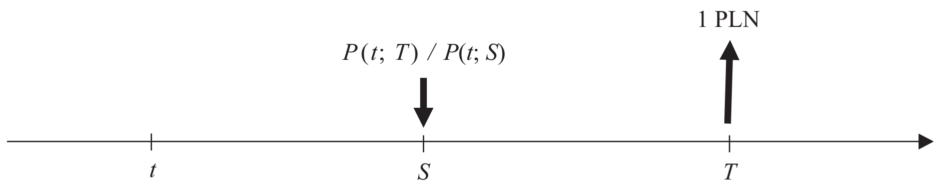

Figure 1. Cash flows for transactions on bonds maturing at *S* and *T*

Different types of forward interest rates

Simple forward rate:

$$F(t;S,T)=\frac{P(t,S)-P(t,T)}{(T-S)P(t,T)}. \tag{11}$$

Compound forward rate with k compound periods yearly:

$$Y_k(t;S,T)=k\left[\left(\frac{P(t,S)}{P(t,T)}\right)^{1/k(T-S)}-1\right]. \tag{12}$$

Continuous forward rate:

$$R(t;S,T)=-\frac{\ln P(t,T)-\ln P(t,S)}{T-S}. \tag{13}$$

The difference between spot rates and forward rates is the following. Spot rates are for investments that start “now”. Forward rates refer to investments that will be done in the future, but the interest rate for the investment is settled “now”. Figure 2 illustrates this graphically for continuous rates.

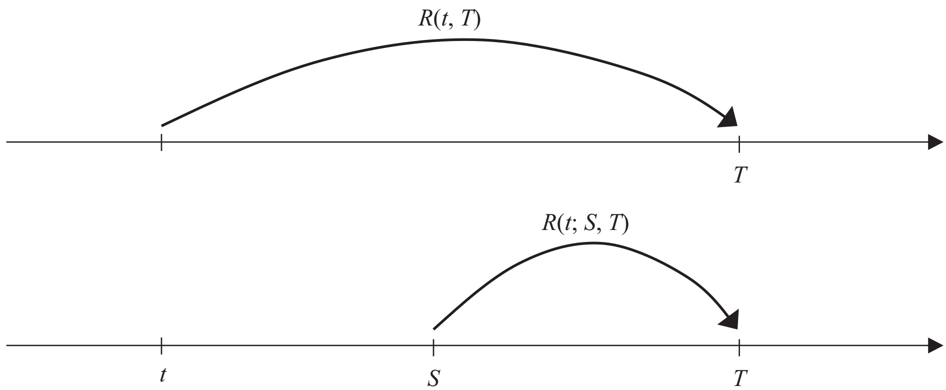

Figure 2. Difference between spot rates and forward rates

Example 2

Suppose that the interest rate for one year from now (with semiannually compounding) is $Y_2(0,1)=10\%$ and the interest rate for one and a half year is $Y_2(0,1.5)=10.3\%$. The discount factors, calculated with the equation (4) are:

$$P(0,1)=\left(1+\frac{0.1}{2}\right)^{-2\cdot 1}=0.907029,$$

$$P(0,1.5)=\left(1+\frac{0.103}{2}\right)^{-2\cdot 1.5}=0.860146.$$

The forward rates of different types for the period starting in one year and ending after one and a half year are:

$$F(0;1,1.5)=\frac{0.907029-0.860146}{0.5\cdot 0.860146}=0.1090=10.90\%,$$

$$Y_2(0;1,1.5)=2\left[\left(\frac{0.907029}{0.860146}\right)^{\frac{1}{2\cdot 0.5}}-1\right]=0.1090=10.90\%,$$

$$R(0;1,1.5)=-\frac{\ln 0.860146-\ln 0.907029}{0.5}=0.1061=10.61\%.$$

All forward rates defined by equations (11)-(13) apply over discrete time interval. In general, forward rates can be calculated for time interval of any length. One can consider what will happen if the interval becomes shorter and shorter. **Instantaneous forward rates** are forward rates obtained when the length of interval becomes infinitesimally small. Mathematically, they are derived as a limit of forward rates with $S \to T$. A special case of instantaneous forward rate is a **short rate** – a rate for interval starting now (i.e. $T = t$). Calculating appropriate limits in equations (11)-(13) we obtain the following definitions.

Instantaneous forward rate and short rate
Instantaneous forward rate equals

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{14}$$

Short rate:

$$r(t) = f(t,t) = \lim_{T \to t} R(t,T). \tag{15}$$

The instantaneous forward rate is a forward rate for infinitely small time interval starting at T. One can interpret it as the marginal cost of borrowing for that moment of time. The short rate is the marginal cost of borrowing now. For any moment t the rates $f(t,T)$ expressed as a function of maturity time form a **term-structure of forward years** or **forward curve**.

By integrating equation (14) we obtain the following formula for discount factors expressed with forward rates:

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds \right). \tag{16}$$

Using equations (10) and (13) the following formulas can be derived:

$$R(t,T) = \frac{1}{T-t} \int_t^T f(t,s)ds, \tag{17}$$

$$R(t;S,T) = \frac{1}{T-S} \int_S^T f(t,s)ds. \tag{18}$$

According to (17) and (18) continuous rate (spot and forward) is the average of instantaneous forward rated over the interval of investment.

Exercise 1. Derive the equation (8)-(10).

Exercise 2. Derive the formulae (11)-(13).

Exercise 3. Assume that the simple spot rate for 9 month from now is 12%. Calculate the discount factor, continuous rate and compound rate with quarterly compounding.

Exercise 4. Simple spot rates for 3, 6, 9 and 12 months are 5.3%, 5.4%, 5.25% and 5.2%, respectively. Compute forward rates for the periods starting in 3 months and ending in 6 month, starting in 6 months and ending in 9 months, and starting in 9 months, ending after a year.

5.2. Estimating term structure of interest rates

In theoretical framework it is assumed that there exists a zero-coupon bond for any maturity and the discount factor is equal to the current price of this bond. This can be seen as a crude approximation of the reality, as in real markets there are usually only zero-coupon bonds for short maturities. Bonds with longer maturities usually bear coupons. The discount factors for many maturities should be estimated. Estimation of the term structure requires fitting various functions to observed interest rates and bonds' prices. The functional form should be flexible enough to capture stylized facts concerning the shape of the term structure. It is believed that estimation method should have the following characteristics:

1. It should be suitable to fit various of data.
2. Usually it is assumed that the estimated interest rates for all maturities should be positive.
3. The estimated discount factors, spot rates and forward rates should change continuously and smoothly with maturities.
4. The estimated spot and forward rates should converge asymptotically towards a constant, which is equal to the consol rate, i.e. the rate for very long (infinitely long) maturities.

5.2.1. Data for estimation

5.2.1.1. Interbank market interest rates

Interbank rates are rates at which deposits between commercial banks are exchanged. There are two types of rates: *bid* rates: the rates which banks are willing to pay for deposits from other banks, and *offer* rates: the rates at which banks can borrow unsecured funds from other banks. The maturities range from overnight (deposit is closed in the next day) to one year.

The most important interbank rates are LIBOR (London Interbank Offer Rate) in the London interbank market. LIBOR rates are calculated for five currencies and seven maturities and and since 2014 are calculated by ICE Benchmark Administration (IBC) and published on each business day at 11 a.m. by the Thompson Reuters. LIBOR is used as a key reference rate for a variety of global financial instruments. The rates are calculated based on the survey among a panel of major banks

In the euro-zone the most important interbank rates are EURIBOR rates, published by the European Money Markets Institute at 11 a.m. CET. As in the case of LIBOR, it is calculated on the basis of a survey in which 20 banks are asked at which rate they are willing to give unsecured loans in euro. They are not "real" transaction rates. On each business day European Central Bank at 7 p.m. CET publishes EONIA rate–effective overnight rate in the interbank rate. It is computed as a weighted average of all overnight unsecured lending transactions in the interbank market, undertaken in the European Union and European Free Trade Association countries.

In Poland the rates in the interbank market are measured by WIBOR (Warsaw Interbank Offer Rate) rates. They are calculated by the Warsaw Stock Exchange[1], based on the survey among banks, and are published at 11 a.m. CET. Table 1 contains exemplary WIBOR rates on 26 January 2018.

Table 1. WIBOR rates on 26 January 2018

Maturity	Name	Rate (%)
1 day	WIBOR ON	1.15
1 week	WIBOR 1W	1.54
1 month	WIBOR 1M	1.65
3 months	WIBOR 3M	1.72
6 months	WIBOR 6M	1.81
1 year	WIBOR 1Y	1.85

[1] Since 30 June 2017. Earlier for almost 25 years WIBOR rates had been administrated by ACI Polska.

Apart from WIBOR rates, also a weighted average of overnight transaction rates is calculated and published at 5 p.m. CET as the POLONIA rate. On 26 January 2018 the POLONIA rate was 0.85% – thirty basis points below WIBOR ON.

In many other European and non-European countries there are available indicators concerning rates in the local interbank market. There exists, for example, STIBOR (Sweden), TIBOR rates (Japan), SIBOR (Singapore), MIBOR (India).

5.2.1.2. Bonds

The main part of fixed-income investments are associated with various kind of bonds. A bond is a debt contract in which an issuer promises to pay a stream of cash flows over a fixed time horizon. In case of zero-coupon bonds the stream consists of a single payment–return of the debt in the time of maturity. This kind of bonds usually has short maturity: up to two years. Bonds with longer maturities bring interest, which is paid periodically until the maturity of the bond. These interest payments are called **coupons** and these kind of bonds are called **coupon bonds**. At maturity the debt is payed off and the bond brings the **face value** (also called the **par value** or **principal**).

From purely financial point of view, a bond is a stream of cash flows, as it is illustrated in Figure 3. Usually the payments before maturity are interest paid according to the fixed **coupon rate**. The last payment equals interest plus principal. Thus

$$C_k = \frac{c}{m}V \quad \text{for } k = 1, \ldots, n-1,$$

$$C_n = \frac{c}{m}V + V,$$

where c is the coupon rate, coupons are paid m times a year and V is the face value of the bond. For example, for a bond with the face value of PLN 1000 and coupon rate of 5% with coupons paid semiannually, all payments except for the last one equal PLN 25 and the last payment is PLN 1025. The structure of coupon payments

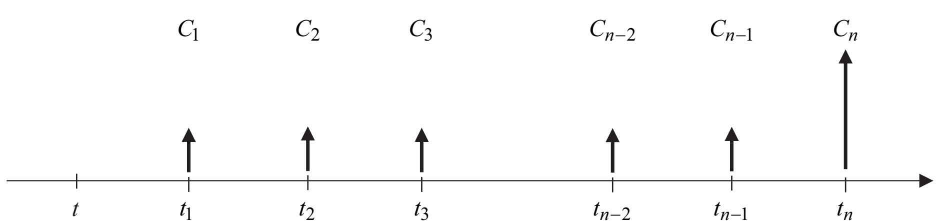

Figure 3. Coupon bond as a series of cash flows

with a constant rate is considered standard for bonds. However, there can be bonds with a different pattern of payments. Therefore we assume only that payments C_1, …, C_n can be arbitrary, but they are known at the initial time t.

The price of a bond is equal to the current value of the cash flow that it generates. Formally:

$$B(t)=C_1P(t,t_1)+C_2P(t,t_2)+\ldots+C_nP(t,t_n)=\sum_{k=1}^{n}C_kP(t,t_k), \tag{19}$$

where $B(t)$ denotes the price of the bond. Assuming continuous discounting this equation can be also rewritten as

$$B(t)=\sum_{k=1}^{n}C_ke^{-R(t,t_k)\cdot(t_k-t)}. \tag{20}$$

Thus market prices of bonds contain information about discount factors and interest rates. The estimation of term structure of interest rates requires extracting this information from observed prices.

The **yield to maturity** (or shorter **yield**) is the discount rate that makes the sum of the discounted values of cash flows of the bond equal to current price of the bond. It is an internal rate of return of all cash flows connected with the bond, including the expenditure for buying it. Assuming continuous discounting, the yield to maturity, y, is the solution of the equation

$$B(t)=C_1e^{-y(t_1-t)}+C_2e^{-y(t_2-t)}+\ldots+C_ne^{-y(t_n-t)}=\sum_{k=1}^{n}C_ke^{-y(t_k-t)}. \tag{21}$$

The yield to maturity can be seen as a weighted average of zero-coupon rates. It depends on the current market price of the bond. If the price is equal to its face value, then the yield equals the coupon rate. When the price is lower than the face value, the yield is higher than the coupon rate and in the opposite situation the yield is lower than the coupon rate.

Example 3

Consider a bond which will pay a coupon of 10 PLN in one year and a coupon with the principal in two years, which gives a sum of 110 PLN. The current price of the bond at moment $t=0$ is 98.1 PLN. The equation for calculating yield is

$$98.1 = 10e^{-y} + 110e^{-2y}.$$

Defining $x = e^{-y}$ yields a quadratic equation:

$$110x^2 + 10x - 98.1 = 0.$$

The equation has two solutions $x_1 = -0.99$ and $x_2 = 0.9$, but we consider only the positive one. The yield to maturity of the bond is

$$y = -\ln(x_2) = -\ln(0.9) = 0.1054 = 10.54\%.$$

The price of a bond and its yield to maturity are connected with each other. Since the yield is usually seen as a measure of general profitability of investments in a bond, one can ask how the price changes with the changes in the yield. The traditional measure for the sensitivity of bond's price to changes in the yield is duration. In the following definitions we assume that $t = 0$.

Duration of the bond is defined as[5]

$$D = -\frac{1}{B}\frac{\partial B}{\partial y} = \frac{1}{B}\sum_{k=1}^{n} t_k C_k e^{-t_k y}. \tag{22}$$

The duration can be used to approximate the percentage change in the bond price when yield changes:

$$\frac{\Delta B}{B} \cong -D\Delta y.$$

Example 4

Consider a five year bond with the face value of PLN 100 with the coupon rate of 5% and coupons payed yearly. Assume that the bond's yield to maturity is 6%. The calculations of modified duration of the bond is given in Table 2.

The price of the bond is $B = 95.0388$ PLN and its duration equals $D = \frac{430.7692}{95.0388} = 4.5326$. It means that when the yield to maturity of the bond in-

[2] When assuming non-continuous compounding one can define two types of duration: Maculay's duration and modified duration. By continuous compounding, as we assumed here, those two parameters are equal. You can find more information for example in chapter 4.8 (Hull, 2015).

Table 2. Calculation of the duration

t_k	C_k	t_kC_k	$C_ke^{-yt_k}$	$t_kC_ke^{-yt_k}$
1	5	5	4.7088	4.7088
2	5	10	4.4346	8.8692
3	5	15	4.1764	12.5291
4	5	20	3.9331	15.7326
5	105	525	77.7859	388.9296
			95.0388	**430.7692**

creases by 1 percentage point (p.p.), its price drops approximately by 4.5326%. On the other hand, when the yield decreases by 1 p.p., the price rises by 4.5326%. In fact, when the yield changes from 6% to 7%, the price of the bond will be equal to 90.8329 PLN, thus the price will decrease by 4.426%. Thus the duration a good approximation of the price changes.

Bonds' prices quoting conventions

Newly issued bonds are sold by an issuer on the primary market. Afterwards they are traded on secondary markets (usually some stock exchanges), where investors can buy them from private persons or institutions, which are willing to sell them. The transaction prices of zero-coupon bonds and coupon bonds are quoted using different conventions. On many markets zero-coupon bonds are quoted using discount rates calculated on the basis of their actual prices. Prices of coupon bonds (but also zero-coupon ones, as for example in Poland) are quoted as a percentage of the principal value. However, the quoted prices (the so-called **clean prices**) are not the prices of transactions (**dirty prices**). The real transaction prices are diminished by **accrued interest**, i.e. the value of due interest, which are calculated according to the number of days since the last coupon payment.

Example 5

On 1st December the quoted price of a bond was 102.15. The bond has the face value of PLN 1000 and its coupon rate is 6%. Coupons are paid twice a year: on 2nd January and 2nd July. Since the last coupon payment 152 days have passed. The yearly coupon is PLN 60 and the coupon paid in one-coupon period is PLN 30. The length of the current coupon period (from 2nd July to 2nd January) is 184 days. The accrued interest for the date 1st December is thus $\frac{152}{184}\cdot 30 = 24.78$. The real price in transaction was $1021.5 + 24.78 = 1046.28$ PLN.

Exercise 5. Consider a five year bond with the face value of PLN 100 and the coupon rate of 6% paid semiannually. The current price of the bond is PLN 106.70. Calculate the yield to maturity of the bond and its duration. For the first task use Excel function IRR.

Exercise 6. Calculate the duration of a zero-coupon bond maturing in two years. Its face value is PLN 100 and current price is PLN 92.31.

5.2.2. Bootstrapping

The bootstrapping is the simplest method of estimating the term structure. The idea is similar to obtaining interest rates from prices of zero-coupon bonds. The method consists of iteratively extracting discount factors from prices of a sequence of coupon bonds with increasing maturity.

Assume, for example, that we know the price of a zero-coupon bond maturing in three months. Let us denote it by $B(0.25)$. For notional simplicity, we assume that today's date is $t=0$. The discount factor for three months from now equals

$$P(0.25)=\frac{B(0.25)}{C_1^1},$$

where C_1^1 is the only payment of the bond – its face value. Let $B(0.5)$ be the price of the bond with two payments left. The first of these payments, C_1^2 is in three months, and the second one, C_2^2, is in six months. The price equals

$$B(0.5)=C_1^2 P(0.25)+C_2^2 P(0.5).$$

Hence we can calculate the discount factor for six months:

$$P(0.5)=\frac{B(0.5)-C_1^2 P(0.25)}{C_2^2}.$$

Similarly, if we know the price of a bond maturing in nine months with payments (C_1^3, C_2^3 and C_3^3) in three, six and nine months, then we can calculate the discount factor for nine months:

$$P(0.75)=\frac{B(0.75)-C_1^3P(0.25)-C_2^3P(0.5)}{C_3^3}.$$

If we have an appropriate set of bonds with longer and longer maturities, we can continue this procedure and compute discount factors for longer terms. Instead of calculating discount factors sequentially, we can also obtain the solution more directly, using linear algebra. Suppose that we have a set of *K* bonds that have payments in moments t_1, t_2, …, t_K . Let C_i^k be the total cash flow on *k* th bond on the date t_i and B^k be the price of bond *k*. The prices of all bonds are given by the equation:

$$B=\begin{pmatrix} B^1 \\ B^2 \\ \vdots \\ B^K \end{pmatrix}=\begin{bmatrix} C_1^1 & C_2^1 & \cdots & C_K^1 \\ C_1^2 & C_2^2 & \cdots & C_K^2 \\ \vdots & \vdots & \ddots & \vdots \\ C_1^K & C_2^K & \cdots & C_K^K \end{bmatrix}\begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_K) \end{pmatrix}=CF\begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_K) \end{pmatrix},$$

where *B* is the vector with bonds' prices and *CF* is the matrix with cash flows (i.e. the matrix in which the row *k* consists of cash flows of bond *k*). The discount factors for maturities t_1, t_2, …, t_K can be obtained by multiplying both sides by the inverse of the cash flow matrix. Thus

$$\begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_K) \end{pmatrix}=CF^{-1}B. \tag{23}$$

If we have a set of bonds with the structure of payments as described above (a bond with one payment left, a bond with two payments, etc.), then the cash flow matrix CF has the following structure:

$$CF=\begin{bmatrix} C_1^1 & 0 & 0 & \cdots & 0 \\ C_1^2 & C_2^2 & 0 & \cdots & 0 \\ C_1^3 & C_2^3 & C_3^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_1^K & C_2^K & C_3^K & \cdots & C_K^K \end{bmatrix}.$$

Example 6

In Table 3 there are data concerning four bonds. We assume that all bonds pay coupons yearly.

Table 3. Data for the bootstrapping method example

Bond	Price	Maturity (years)	Coupon rate (%)
1	97.90	1	5.0
2	98.70	2	5.5
3	102.15	3	6.0
4	96.00	4	5.0

The cash flow matrix for these bonds is given by

$$CF = \begin{bmatrix} 105 & 0 & 0 & 0 \\ 5.5 & 105.5 & 0 & 0 \\ 6 & 6 & 106 & 0 \\ 5 & 5 & 5 & 105 \end{bmatrix}$$

and the vector of prices is

$$B = \begin{pmatrix} 97.90 \\ 98.70 \\ 102.15 \\ 96.00 \end{pmatrix}.$$

Using equation (23) we obtain discount factors for maturities range from one year to four years:

$$\begin{pmatrix} P(1) \\ P(2) \\ P(3) \\ P(4) \end{pmatrix} = \begin{bmatrix} 105 & 0 & 0 & 0 \\ 5.5 & 105.5 & 0 & 0 \\ 6 & 6 & 106 & 0 \\ 5 & 5 & 5 & 105 \end{bmatrix}^{-1} \begin{pmatrix} 97.90 \\ 98.70 \\ 102.15 \\ 96.00 \end{pmatrix} = \begin{pmatrix} 0.932394 \\ 0.886920 \\ 0.860708 \\ 0.786628 \end{pmatrix}.$$

The interest rates can be calculated using equations (8)-(10). For example, assuming continuous compounding we obtain

$$R(1) = \ln P(1) = -\ln 0.932394 = 0.0700 = 7\%,$$

$$R(2) = \frac{1}{2}\ln P(2) = -\frac{1}{2}\ln 0.886920 = 0.0600 = 6\%,$$

$$R(2) = \frac{1}{3}\ln P(2) = -\frac{1}{3}\ln 0.860708 = 0.0500 = 5\%,$$

$$R(4) = \frac{1}{4}\ln P(4) = -\frac{1}{4}\ln 0.786628 = 0.0600 = 6\%.$$

5.2.3. Spline method

The estimation of term structure based on splines consists in dividing the term structure into many segments using a series of so-called **knot points**. Then one uses different functions from the same class (for example polynomials or exponential functions) to describe term structure over these segments. The class of functions is constrained to be continuous and smooth around all the knot points to ensure continuity and smoothness of fitted curves.

A spline of the discount function is defined by the following equation:

$$P(T) = 1 + \sum_{j=1}^{n} \alpha_j g_j(T), \tag{24}$$

where g_1, g_2, …, g_n are functions that form the basis of the spline and α_1, α_2, …, α_n are parameters that should be estimated. Since the discount factor for moment $T = 0$ is 1, thus $g_j(0) = 0$ for all $j = 1, \ldots, n$.

As for the set of functions g_1, g_2, …, g_n, among practitioners very popular is the method used by McCulloch, who proposed to use cubic polynomials. The **McCulloch cubic splines** are defined as follows. The horizon of all maturities is divided into $n-2$ intervals by $n-1$ knot points: $\xi_1 < \xi_2 < \ldots < \xi_{n-1}$, where $\xi_1 = 0$. We also set $\xi_0 = \xi_1 = 0$ and define ξ_n as the end of the time horizon (the longest maturity of the bonds in the dataset). The basis function is defined by

$$g_j(T)=\begin{cases} 0 & \text{for } T<\xi_{j-1} \\ \dfrac{\left(T-\xi_{j-1}\right)^3}{6(\xi_j-\xi_{j-1})} & \text{for } \xi_{j-1}\le T<\xi_j \\ \dfrac{\left(\xi_j-\xi_{j-1}\right)^2}{6}+\dfrac{\left(\xi_j-\xi_{j-1}\right)\left(T-\xi_j\right)}{2}+\dfrac{\left(T-\xi_j\right)^2}{2}+ & \\ -\dfrac{\left(T-\xi_j\right)^3}{6(\xi_{j+1}-\xi_j)} & \text{for } \xi_j\le T<\xi_{j+1} \\ (\xi_{j+1}-\xi_{j-1})\left(\dfrac{2\xi_{j+1}-\xi_j-\xi_{j-1}}{6}+\dfrac{T-\xi_{j+1}}{2}\right) & \text{for } T\ge\xi_{j+1} \end{cases} \tag{25}$$

for $j=1,\ldots,n-1$ and

$$g_n(T)=T. \tag{26}$$

Figure 4 contains a graph of the first four splines for $n=4$.

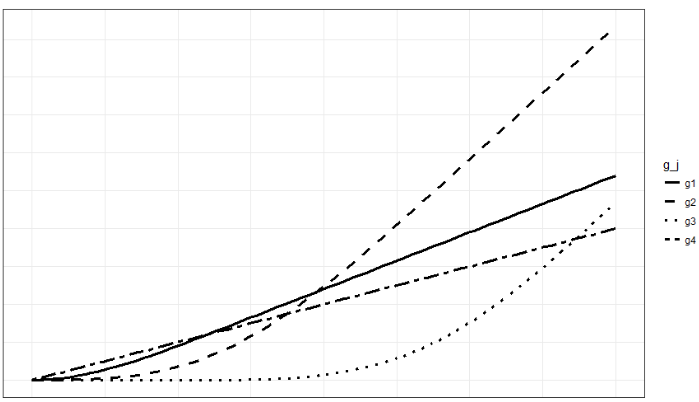

Figure 4. First four McCulloch cubic splines

Assume that we have a set of K bonds and their actual prices are B^1, B^2, …, B^K. The payments for bonds are in the moments t_1, t_2, …, t_m and let C_i^k be the total cash flow on k th bond on the date t_i (if the bond k brings no payment at the moment t_i, i.e. because this moment falls after the maturity of the bond then $C_i^k=0$). The market price of the bond k is

$$B^k = \sum_{i=1}^{m} C_i^k P(t_i) + \varepsilon_k \tag{27}$$

where ε_k is an idiosyncratic error for the bond k – its deviation from the "true" price. Substituting (24) into (27), we obtain

$$B^k = \sum_{i=1}^{m} C_i^k \left(1 + \sum_{j=1}^{n} \alpha_j g_j(t_i)\right) + \varepsilon_k.$$

By rearranging the terms, we obtain

$$B^k - \sum_{i=1}^{m} C_i^k = \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{n} C_i^k g_j(t_i) + \varepsilon_k. \tag{28}$$

Let us define $Y^k = B^k - \sum_{i=1}^{m} C_i^k$ and $X_j^k = \sum_{i=1}^{n} C_i^k g_j(t_i)$. The equation (28) can be rewritten as

$$Y^k = \sum_{j=1}^{m} \alpha_j X_j^k + \varepsilon_k = \alpha_1 X_1^k + \alpha_2 X_2^k + \ldots + \alpha_n X_n^k + \varepsilon_k. \tag{29}$$

Equation (29) is a linear regression of the variable Y against variables X_1, X_2, …, X_n and the parameters α_1, …, α_n can be obtained using standard econometric methods – for example with ordinary least squares (OLS) estimation.

Example 7

Table 4 contains data concerning five bonds with maturities varying from one year to six years. We assume that coupons are payed annually.

Table 4. Data for the cubic splines estimation

Bond	Price	Maturity (years)	Coupon rate (%)
1	99.46	1	3.0
2	98.92	2	3.5
3	97.02	3	3.0
4	94.36	4	3.0
5	101.72	5	3.5
6	94.93	6	4.0

We take three knot points ($n=3$): $\xi_1 = 0$, $\xi_2 = 3$ and $\xi_3 = 6$. Thus we will use three base functions g_1, g_2 and g_3. Table 5 contains values of these function for the moments of coupon payments.

Table 5. The values of the functions g_j

t_i	g_1	g_2	g_3
1	0.4444	0.0556	1.0000
2	1.5556	0.4444	2.0000
3	3.0000	1.5000	3.0000
4	4.5000	3.4444	4.0000
5	6.0000	6.0556	5.0000
6	7.5000	9.0000	6.0000

The next step is to calculate the values of the variables Y, X_1, X_2 and X_3 for all bonds. Table 6 contains exemplary calculations for the bond maturing in four years.

Table 6. The values of the functions g_j

i	C_i^4	$C_i^4 g_1(t_i)$	$C_i^4 g_2(t_i)$	$C_i^4 g_3(t_i)$
1	3.0	1.3333	0.1667	3.0
2	3.0	4.6667	1.3333	6.0
3	3.0	9.0	4.5	9.0
4	103.0000	463.5000	354.7778	412.0
5	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0
	112.0	478.5	360.7778	430.0

The last row contains sums of all numbers in the given column. We have thus $Y^4 = 94.36 - 112 = -17.64$, $X_1^4 = 478.5$, $X_2^4 = 360.778$ and $X_3^4 = 430$. Performing similar computations for all the bonds we obtain values of all the variables, summarized in Table 7.

Table 7. The values of variables in the regression

k	Y^k	X_1^k	X_2^k	X_3^k
1	–3.54	45.78	5.72	103
2	–8.08	162.56	46.19	210.5
3	–11.98	315	156	318
4	–17.64	478.5	360.78	430
5	–15.78	654.25	645.81	552.5
6	–29.07	842	982	684

Performing linear regression on the variables from Table 7, one obtain the following function describing discount factors:

$$P(T)=1+0.019393g_1(T)+0.008904g_2(T)-0.050138g_3(T).$$

Continuous interest rates $R(T)$ can be calculated as $R(T)=-\ln P(T)/T$.

5.2.4. Nelson-Siegel model

Nelson and Siegel used an exponential function in describing forward rates over the whole maturity range. They proposed the following functional form to describe instantaneous forward rates:

$$f(T)=\alpha_1+\alpha_2 e^{-T/\beta}+\alpha_3\frac{T}{\beta}e^{-T/\beta}. \tag{30}$$

The advantage of this method lies in the fact that the function has an asymptotical value for $T\to\infty$. For this reason it is preferred by many practitioners. The continuous rates consistent with the forward rates given above can be calculated from equation (17):

$$R(T)=\alpha_1+(\alpha_2+\alpha_3)\frac{\beta}{T}\left(1-e^{-\frac{T}{\beta}}\right)-\alpha_3 e^{-\frac{T}{\beta}}. \tag{31}$$

By equation (6) the discount factors are given by the following equation:

$$P(T)=\exp\left[-\alpha_1 T-\beta(\alpha_2+\alpha_3)\left(1-e^{-\frac{T}{\beta}}\right)+\alpha_3 T e^{-\frac{T}{\beta}}\right]. \tag{32}$$

Figure 5 presents exemplary terms structure of both spot and forward rates in the Nelson-Siegel model. The curves are typical for this model. The lines have one hump that can be directed either downward (as in the graph) or upward. Then the rates converge to their asymptotic value.

The model has four parameters: α_1, α_2, α_3 and β. The parameters have interpretation.

- α_1 is the consol rate (i.e. rate for very long maturities). It is asymptotic value for both spot and forward rates: $f(\infty)=R(\infty)=\alpha_1$.

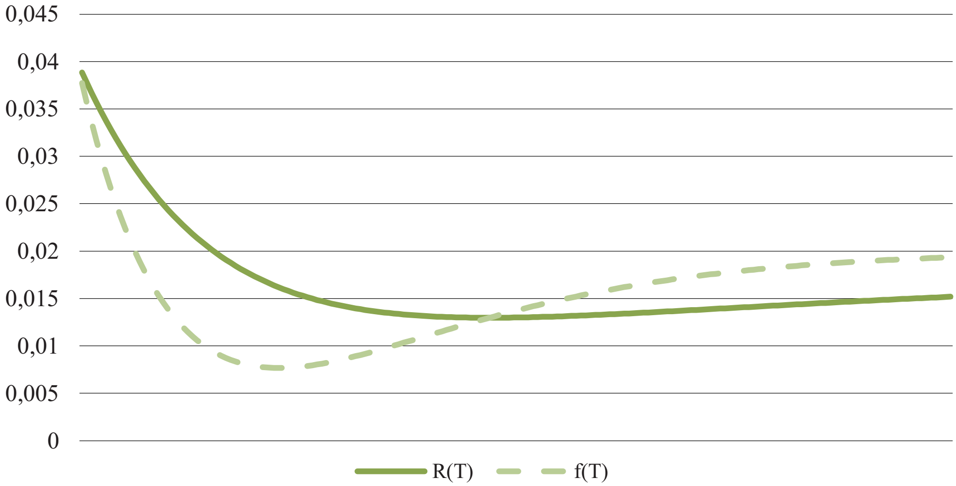

Figure 5. Exemplary term structure in the Nelson-Siegel model

- The short rate is $\alpha_1+\alpha_2$: $r(0)=f(0)=R(0)=\alpha_1+\alpha_2$. Thus the spread between consol rate and short rate is $-\alpha_2$.
- The parameter α_3 is responsible for the shape of the term structure. When $\alpha_3>0$, the term structure attains the maximum value and its shape is concave (hump directed upward). When $\alpha_3<0$, the term structure attains the minimum value and its shape is convex (hump directed downward, as in Figure 5).
- The parameter $\beta>0$ is the speed of convergence of the terms structure toward the consol rate. The lower its value is, the quicker interest rates approach their asymptotic value.

According to equation (30) the shape of the term structure can be expressed as a combination of three factors. The first one, α_1, determines the overall level of interest rates (the height of the curve). The second factor, $\alpha_2 e^{-T/\beta}$, determines how interest rates change with the changes in maturity (the slope of the curve). The third factor, $\alpha_3 \frac{T}{\beta} e^{-T/\beta}$, is responsible for the hump in the curve. Figure 6 illustrates the shapes of these factors.

The estimation of the model consists of choosing appropriate values of the parameters. This proceeds as follows. Suppose that we have a set of K bonds with cash flows at moments t_1, t_2, …, t_M. Let C_m^k be a payment of bond k at moment t_k. The theoretical price of the bond k is

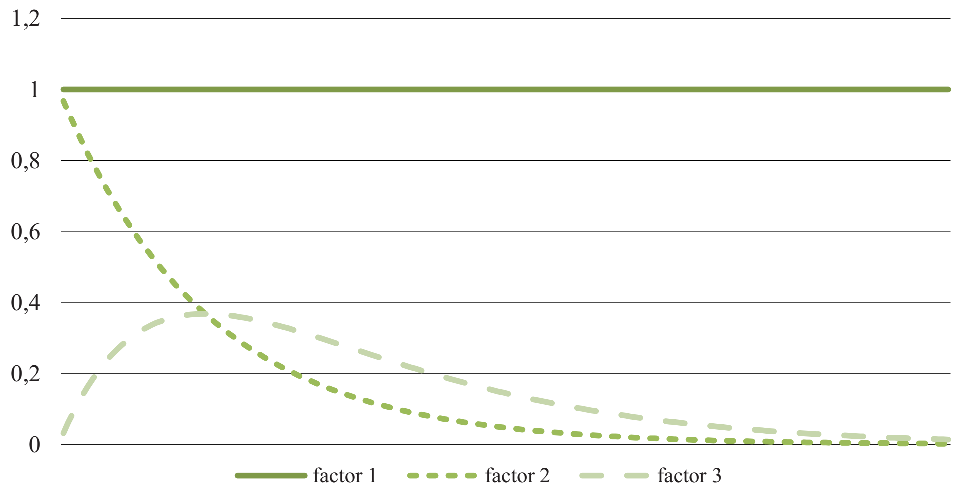

Figure 6. Three factors determining term structure of forward rates in the Nelson--Siegel model

$$\hat{B}^k = \sum_{m=1}^{M} C_m^k \exp\left[-\alpha_1 t_m - \beta(\alpha_2+\alpha_3)\left(1-e^{-\frac{t_m}{\beta}}\right)+\alpha_3 t_m e^{-\frac{t_m}{\beta}}\right].$$

The market prices of these bonds are B^1, B^2, ..., B^K. The parameters α_1, α_2, α_3 and β are estimated by minimizing the sum of squared differences between theoretical and observed prices, i.e. by solving the following optimization problem:

$$\min F(\alpha_1, \alpha_2, \alpha_3, \beta) = \sum_{k=1}^{K} \left(\hat{B}^k - B^k\right)^2, \tag{33}$$

subject to the constrains:

$$\alpha_1, \alpha_1 + \alpha_2, \beta > 0\,. \tag{34}$$

The problem (33)-(34) is a problem of nonlinear optimization. It can be solved only numerically, using appropriate algorithms. For such algorithms to find the optimal solution it is crucial to appropriately set the starting values of the parameters. The interpretation of the parameters gives some advice in this case. The starting value for the parameters α_1 and α_2 can be obtained from the yields of the bonds with the shortest and longest times to maturity. The starting values of the parameters α_3 and β can be established by analyzing the structure of yields with different maturities.

The problem (33)-(34) is the simplest approach to estimating term structure. It is sometimes claimed that when minimizing the unweighted price errors, bonds with longer maturity obtain a higher weighting, because their prices are more sensitive to changes in interest rates. To avoid this problem one can try to minimize weighted sum of squared errors. In this case the optimization problem has the following form:

$$\min F(\alpha_1, \alpha_2, \alpha_3, \beta) = \sum_{k=1}^{K} w_k \left(\hat{B}^k - B^k\right)^2 \tag{35}$$

with constrains given by (34). Usually the weights w_k are based on the inverse of duration of the bonds, i.e.

$$w_k = \frac{D_k^{-1}}{\sum_{i=1}^{K} D_i^{-1}},$$

where D_k is the duration of bond *k*.

5.2.5. Svensson model

The Svensson model is an extension of the Nelson-Siegel model. As in the later one, the exponential function is used. Instantaneous forward rates for different maturities are described by the following function:

$$f(T) = \alpha_1 + \alpha_2 e^{-T/\beta_1} + \alpha_3 \frac{T}{\beta_1} e^{-T/\beta_1} + \alpha_4 \frac{T}{\beta_2} e^{-T/\beta_2}. \tag{36}$$

Comparing this with the equation in the Nelson-Siegel model, one term is added. This term allows for the second hump in the curve describing interest rates. Just as the Nelson-Siegel model, the function in the Svensson model has an asymptotical value for $T \to \infty$, which makes the model useful in practical applications. The continuous rates consistent with the forward rates given above can be calculated from equation (17):

$$R(T)=\alpha_1+\alpha_2\frac{\beta_1}{T}\left(1-e^{-\frac{T}{\beta_1}}\right)+\alpha_3\left[\frac{\beta_1}{T}\left(1-e^{-\frac{T}{\beta_1}}\right)-e^{-\frac{T}{\beta_1}}\right]+$$
$$+\alpha_4\left[\frac{\beta_2}{T}\left(1-e^{-\frac{T}{\beta_2}}\right)-e^{-\frac{T}{\beta_2}}\right]. \quad (37)$$

Discount factors are thus given by the equation:

$$P(T)=\exp\left\{-\alpha_1 T-\alpha_2\beta_1\left(1-e^{-\frac{T}{\beta_1}}\right)-\alpha_3\left[\beta_1\left(1-e^{-\frac{T}{\beta_1}}\right)-Te^{-\frac{T}{\beta_1}}\right]+\right.$$
$$\left.-\alpha_4\left[\beta_2\left(1-e^{-\frac{T}{\beta_2}}\right)-Te^{-\frac{T}{\beta_2}}\right]\right\}. \quad (38)$$

Figure 7 presents exemplary terms structure of both spot and forward rates in the Svensson model. The lines can have two humps: directed downward or upward. Then the rates converge to their asymptotic value.

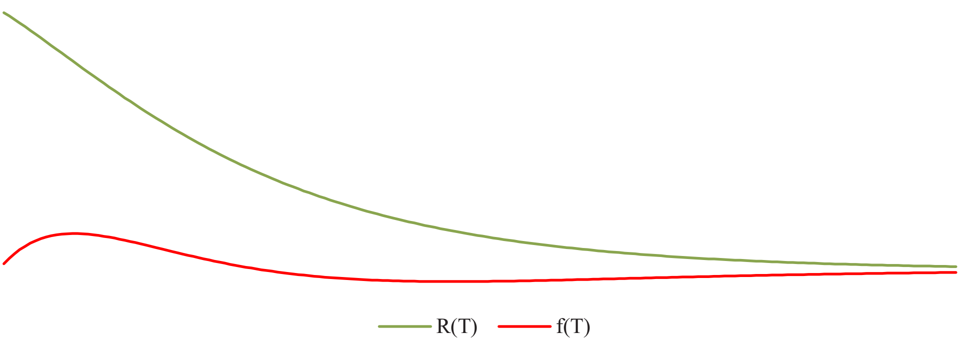

Figure 7. Exemplary term structure in the Svensson model

The model has six parameters: two more than in the Nelson-Siegel model. The interpretations of the parameters α_1, α_2, α_3 and β_1 are the same as in the previous model. The parameters α_4 and β_2 are responsible for the shape of the second hump and their interpretation is following.

- The parameter α_4 is responsible for the shape of the second hump. When $\alpha_4>0$, the hump is directed upward. When $\alpha_4<0$, the hump is directed downward.

- The parameter $\beta_2 > 0$ determines the speed with which the second hump decays to zero and term structure approaches the consol rate. The lower its value is, the quicker the interest rates approach their asymptotic value.

The shape of the term structure can be expressed as a combination of three factors. The first three factors are the same as in the Nelson-Siegel model. The fourth factor, $\alpha_4 \frac{T}{\beta_2} e^{-T/\beta_2}$, is responsible for the second hump in the curve. Figure 8 illustrates the shapes of these factors.

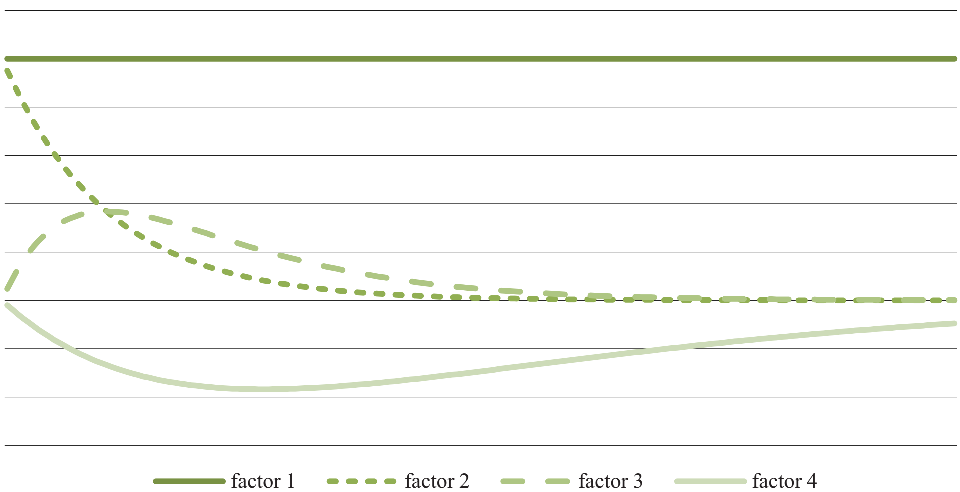

Figure 8. Three factors determining term structure of forward rates in the Svensson model

The estimation procedure is similar to the procedure used for the Nelson-Siegel model. It is based on minimizing differences between the observed prices of bonds and their theoretical prices for given values of parameters. One has to solve the following problem:

$$\min \sum_{k=1}^{K} \left(\hat{B}^k - B^k \right)^2, \tag{39}$$

subject to the constrains:

$$\alpha_1, \alpha_1 + \alpha_2, \beta_1, \beta_2 > 0.$$

5.2.6. Estimation of a term structure by central banks

Many central banks in the world systematically do the computations of the term structure in their countries. The results of their estimation are very often published in their information materials or on their websites. Since 1996 many central banks in the world systematically have been reporting their estimates to the Bank of International Settlements (BIS) and the BIS Data Bank Services provide access to these data. Central banks in different countries use various methods in the estimation of the term structure, but in most cases the contributing central banks adopted one of the parametric methods: either Nelson-Siegel model or Svensson model. In other cases some kinds of splines are used. Table 8 summarizes methods used by some central banks.

Table 8. Estimation of the term structure of interest rates

Country	Estimation method	Minimized error	Maturity range
Belgium	SV or NS	weighted prices	up to 16 years
Canada	splines	weighted prices	3 months to 30 years
Finland	NS	weighted prices	from 1 to 12 years
France	SV or NS	weighted prices	up to 10 years
Germany	SV	yields	from 1 to 10 years
Italy	NS	weighted prices	up to 30 years
Japan	splines	prices	from 1 to 10 years
Norway	SV	yields	up to 10 years
Spain	SV	weighted prices	up to 10 years
Sweden	splines and SV	yields	up to 10 years
Switzerland	SV	yields	from 1 to 30 years
U.K.	splines	yields	up to 30 years
U.S.	splines	weighted prices and prices	up to 10 years

Source: BIS (2005).

In the table NS means Nelson-Siegel method and SV means Svensson model. In some cases instead of minimizing differences between theoretical and observed prices, as in the problems (33) or (35), the differences between theoretical and observed yield are minimized.

Exercise 7. In the Table 9 below there are data concerning 10 different bonds. It is assumed that all bonds pay coupons semi-annually. Estimate the terms structure of interest rates using the following methods:

a) bootstrapping,

b) McCulloch cubic splines,

c) Nelson-Siegel model,
d) Svensson model.

In the cubic splines method assume that knot points are 0, 3, 7 and 10. Calculation for the Nelson-Siegel model and the Svensson model can be done in Excel using package 'Solver'.

Table 9. The data to estimate the term structure

Bond	Price	Maturity (years)	Coupon rate (%)
1	100.44	1	5
2	100.64	2	5.50
3	101.67	3	6
4	101.95	4	6
5	102.87	5	6.50
6	97.4	6	5.50
7	92.92	7	5
8	92.53	8	5
9	90.64	9	4.50
10	91.05	10	4

Exercise 8. Repeat the estimation from the previous example using *R* package 'termstrc'.

Further readings

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CHAPTER 6
YIELD CURVES CONSTRUCTION METHODS: KEY CONCEPTS AND EVOLUTION OF MARKET PRACTICE

6.1. Introduction

The 2007-2009 credit and liquidity crisis, among other things, resulted in a substantial increase in basis spreads quoted on the market between single-currency interest rate derivatives with varying underlying rate tenors (such as Euribor3M, Euribor6M), primarily interest rate swaps, as well as cross currency basis swaps, i.e. instruments which exchange between floating rates in different currencies. Changes in the market were associated with the increased liquidity risk and a higher payment frequency (quarterly instead of semi-annually, for instance) preferred by the majority of financial institutions in times of crisis.

Market changes described above have also led to a "division" of the interest rate market into sub-segments associated with instruments with 1M, 3M, 6M, or 12M rate tenors. These smaller slices of the interest rate market pie mainly differed with respect to liquidity and credit risks and thus reflected the varying perspectives of different market participants.

The condition of the market after the crisis *has forced us to rethink and redefine* methodologies used to price interest rate derivatives, i.e. financial instruments whose price depends on the present value of future interest-rate-linked cash flows. In the following chapters, we present an updated approach towards yield curves construction methods.

6.2. Key definitions

The purpose of this chapter is to introduce key notations and concepts that are used to characterize prices and yields of basic interest rate derivatives. Overall, the chapter provides a solid foundation for a more extensive discussion of interest rate derivatives that we will undertake in the next chapters.

6.2.1. Key notations

Below we present the key notations used.

Time to maturity $T - t$ is the amount of time (expressed in years) from the present time t to the maturity time $T > t$ (Brigo & Mercurio, 2001, p. 4). In case the present time t and the maturity time T are expressed as dates in day / month / year convention, i.e. $D_1 = (d_1, m_1, y_1)$ and $D_2 = (d_2, m_2, y_2)$, one needs to use day count conventions as presented in 6.2.2.

A zero-coupon bond with the maturity time T is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$. It also states that $P(T, T) = 1$ for all T (Brigo & Mercurio, 2001, p. 4). We assume that the zero-coupon bond bears no credit risk.

A spot interest rate is the constant rate at which an investment of $P(t, T)$ unit of currency at time t accrues to yield a unit amount of currency at the maturity time T (Brigo & Mercurio, 2001, p. 6). The simply compounded spot interest rate at time t for the maturity T is formulated as follows:

$$R(t,T) = \frac{1}{T-t} \cdot \frac{1}{P(t,T)}. \tag{1}$$

A forward interest rate prevailing at time t for the period from T_1 to T_2, is an interest rate that can be locked at time t for and investment in a future period with expiry T_1 and maturity T_2, where $t < T_1 < T_2$ (Brigo & Mercurio, 2001, p. 11). The simply compounded forward interest rate at time t for the period from T_1 to T_2 is formulated as follows:

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \cdot \left(\frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)} \right). \tag{2}$$

6.2.2. Day count conventions

Interest rates are always quoted per annum. If one wants to calculate the amount due for the period that is different than one year, it has to be calculated from the quoted rate and the day count fraction for the period. This fraction is calculated by using a day count convention and dividing the counted days by the number of days in a year as assumed with the given convention. Key day count conventions that are used in the market practice are presented below.

Actual/365 According to this convention a year is always 365 days long and the year fraction between two dates is calculated by dividing the actual number of days between them by 365. If by $D_2 - D_1$ we denote the actual number of dates between the two dates $D_1 = (d_1, m_1, y_1)$ included and $D_2 = (d_2, m_2, y_2)$ (excluded), the year fraction is calculated as follows:

$$\frac{D_2 - D_1}{365}. \tag{3}$$

For example, the year fraction between 30th October 2012 and 20th February 2014 is 478/365 = 1.3096.

Actual/360 According to this convention a year is always 360 days long and the year fraction between two dates is calculated by dividing the actual number of days between them by 360. If by $D_2 - D_1$ we denote the actual number of dates between the two dates $D_1 = (d_1, m_1, y_1)$ included and $D_2 = (d_2, m_2, y_2)$ (excluded), the year fraction is calculated as follows:

$$\frac{D_2 - D_1}{360}. \tag{4}$$

For example, the year fraction between 30th October 2012 and 20th February 2014 is 478/360 = 1.3278.

Actual/Actual According to this convention the year fraction between two dates is calculated by dividing the actual number of days between them by actual number of dates in given year (365 or 366). If by $D_2 - D_1$ we denote the actual number of dates between the two dates $D_1 = (d_1, m_1, y_1)$ included and $D_2 = (d_2, m_2, y_2)$ (excluded) and DL as the number of days in leap year and DNL as the number of days in non-leap year, the fraction is calculated as follows:

$$\frac{DL}{366} + \frac{DNL}{365}. \tag{5}$$

For example, the year fraction between 30th October 2012 and 20th February 2014 is 63/366 + 415/365 = 1.3091.

30/360 According to this convention a year is always 360 days long and months are assumed to be 30 days long. If by $D_2 - D_1$ we denote the actual number of dates between the two dates $D_1 = (d_1, m_1, y_1)$ included and $D_2 = (d_2, m_2, y_2)$ (excluded), the year fraction is calculated as follows:

$$\frac{(y_2 - y_1)\cdot 360 + (m_2 - m_1 - 1)\cdot 30 + max(30 - d_1;0) + min(d_2;30)}{360}. \qquad (6)$$

For example, the year fraction between 30th October 2012 and 20th February 2014 is $2\cdot\frac{360}{360} - 8\cdot\frac{30}{360} - \frac{\max(30-30;0)}{360} + \frac{\min(20;30)}{360}$=1.3056.

The maturity time T is also affected by the business day convention. Key business day conventions used in the market practice are presented below.

Following business day convention According to this convention it is assumed that if the maturity date T is not a business day, then it is moved to the following business day. For example, according to the following business day convention for one month accrual period starting 31st July 2013 (Wednesday) and ending 31st August 2013 (Saturday), the accrual period will end on 2nd September 2013 (Monday).

Preceding business day convention According to this convention it is assumed that if the maturity date T is not a business day then it is moved to the preceding business day. For example, according to the preceding business day convention for one-month accrual period starting 31st July 2013 (Wednesday) and ending 31st August 2013 (Saturday), the accrual period will end on 30th August 2013 (Friday).

Modified following business day convention According to this convention it is assumed that if the maturity date T is not a business day, then it is moved to the following business day, unless the day falls within the next month. In this case, the date is moved to the previous working day. For example, according to the modified following business day convention for one-month accrual period starting 31st July 2013 (Wednesday) and ending 31st August 2013 (Saturday), the accrual period will end on 30th August 2013 (Friday). This convention is the most popular in interest rate derivatives market.

End of month business day convention According to this convention it is assumed that if the start date is the last business day of the month, than the maturity date T is also the last business day of the month. In other cases we set the end date by simply adding the desired time period to the start date. For example, according to end of month business day convention for one-month accrual period starting 28th February 2014 (Friday), the accrual period will end on 31st March 2014 (Monday). Whereas for one-month accrual period starting 26th February 2014 (Wednesday), the accrual period will end on 26th March 2014 (Wednesday).

6.2.3. Interest rate derivatives

In this subchapter, we describe the main features of key interest rate derivatives. Instruments described in this chapter are widely used by market participants for speculation and hedging. They also serve as building blocks for bootstrapping algorithms described in the next subchapter.

6.2.3.1. FRA

A forward rate agreement (FRA) is a contract in which two counterparties agree today (T_0) on a fixed interest rate K to be applied for some period (T_1, T_2) in the future (Brigo & Mercurio, 2001, p. 11). In order to minimize the counterparty credit risk, the FRA contract assumes no exchange of the notional. The contract is usually settled at the beginning of the accrual period – T_1, the payoff $CF(FRA)$ is calculated using the following formula:

$$CF(FRA) = N \cdot \frac{\left[L(T_1, T_2) - K\right] \cdot [T_2 - T_1]}{1 + L(T_1, T_2) \cdot [T_2 - T_1]} \tag{7}$$

where:

K – pre-agreed contract fixed rate,
N – contract notional,
$L(T_1, T_2)$ – market reference rate at time T_1 for the period (T_1, T_2).

If:

$L(T_1, T_2) > K$, the buyer (counterparty being long) receives payment calculated using formula (7) from the seller (counterparty being short),
$L(T_1, T_2) < K$, the buyer makes the payment calculated using formula (7) to the seller.

FRA contracts are used in market practice mainly for speculation as they grant exposure to potential changes of short-term money market rates. FRA contracts can also be used as hedging instruments, immunizing the buyer from changes in short-term money market rates.

Derivation of FRA pricing formula was presented by Rebonato (2002, pp. 28-31). FRA fair value at any time t before the maturity date depends on the current level of the estimated forward market reference rate:

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \cdot \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right). \tag{8}$$

By applying formulas (7) and (8) FRA fair value at any time t before the maturity date $FRA(t, T_1, T_2)$ can be calculated by using the following formula:

$$FRA(t, T_1, T_2) = N \cdot \frac{\left[F(t, T_1, T_2) - K\right]}{1 + F(t, T_1, T_2) \cdot \left[T_2 - T_1\right]} \cdot P(t, T_1) \cdot \left[T_2 - T_1\right]. \qquad (9)$$

In practice, when pricing FRA contracts one needs to consider the market conventions that depend on the currency of the contract. Table 1 summarizes market conventions for FRA contracts for key currencies and PLN.

Table 1. Market conventions for FRA contracts for key currencies and PLN

Currency	Market convention
CHF	• Market reference rate: LIBOR CHF 3M, LIBOR CHF 6M • Day count convention: ACT/360
EUR	• Market reference rate: EURIBOR 3M, EURIBOR 6M • Day count convention: ACT/360
GBP	• Market reference rate: LIBOR GBP 3M, LIBOR GBP 6M • Day count convention: ACT/365
USD	• Market reference rate: LIBOR USD 3M, LIBOR USD 6M • Day count convention: ACT/360
PLN	• Market reference rate: WIBOR 1M, WIBOR 3M, WIBOR 6M • Day count convention: ACT/365

6.2.3.2. IRS

An interest rate swap (IRS) is a contract in which two counterparties agree to exchange (swap) two sets of cash flows. In order to minimize the counterparty credit risk, the IRS contract assumes no exchange of the notional. The cash flow scheme for a simple (plain vanilla) IRS contract in PLN is presented on Figure 1.

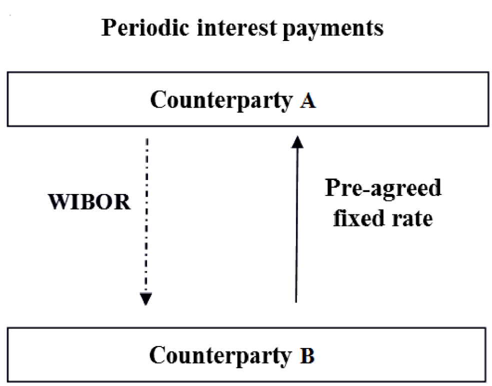

Figure 1. The cash flow scheme for simple (*plain vanilla*) IRS contract in PLN

An IRS contract involves one party making payments calculated using the agreed fixed interest rate (fixed leg), while the other party makes payments calculated using the agreed market reference rate (floating leg). The payments are exchanged on pre-agreed payment dates. However, the frequency of payments for the fixed leg may be different than the frequency of payments for the floating leg. The fixed leg payment on date T_i (CF_fixed) is calculated using the following formula:

$$CF\_fixed(IRS) = N \cdot (T_i - T_{i-1}) \cdot K, \tag{10}$$

where:

N – contract notional,

K – pre-agreed contract fixed rate.

The floating leg payment on date T_i $(CF\_floating)$ is calculated using the following formula:

$$CF\_floating(IRS) = N \cdot (T_i - T_{i-1}) \cdot L(T_{i-1}, T_i), \tag{11}$$

where:

N – contract notional,

$L(T_{i-1}, T_i)$ – market reference rate for the period (T_{i-1}, T_i).

When the fixed leg is paid and the floating leg is received, the IRS is termed Payer IRS (long position in IRS contract). A receiver IRS (short position in IRS contract) functions the other way round (Brigo & Mercurio, 2001, p. 14).

IRS contracts are used in market practice mainly as hedging instruments immunizing the buyer from changes in short-term money market rates.

IRS fair value at any time t before the maturity date can be calculated using the following formula[1]:

$$IRS(t) = PV\_floating(t) - PV\_fixed(t). \tag{12}$$

The fair value of the fixed leg at any time t before the maturity date can be calculated using the following formula:

$$PV\_fixed(t) = N \cdot K \cdot \sum_{i=1}^{L} \left[(T_i - T_{i-1}) \cdot P(t, T_i) \right]. \tag{13}$$

[1] Assuming payer IRS.

where:

L – number of fixed leg payments from time t until the maturity date T_L.

The fair value of the floating leg at any time t before the maturity date can be calculated using the following formula:

$$PV\_floating(t) = N \cdot \sum_{j=1}^{M} \left[\left(T_j - T_{j-1}\right) \cdot F\left(t, T_{j-1}, T_j\right) \cdot P\left(t, T_j\right) \right], \qquad (14)$$

where:

M – number of floating leg payments from time t until the maturity date T_M ($T_L = T_M$).

The formula for forward swap rate calculation can be derived from formula (12). The forward swap rate is defined as the fixed interest rate for IRS contracts that results in contract fair value being zero at time t (Brigo & Mercurio, 2001, p. 15). The forward swap rate can be calculated using the following formula:

$$K\left(t, T_L, T_M, L, M\right) = \frac{\sum_{j=1}^{M} \left[\left(T_j - T_{j-1}\right) \cdot F\left(t, T_{j-1}, T_j\right) \cdot P\left(t, T_j\right) \right]}{\sum_{i=1}^{L} \left[\left(T_i - T_{i-1}\right) \cdot P\left(t, T_i\right) \right]}. \qquad (15)$$

In practice, when pricing IRS contracts, one needs to consider the market conventions that depend on the currency of the contract. Table 2 summarizes market conventions for IRS contracts for key currencies and PLN.

Table 2. Market conventions for IRS contracts for key currencies and PLN

Currency	Floating leg	Fixed leg
CHF	• Day count convention: ACT/360	• Day count convention: 30/360 • Payment frequency: yearly
EUR	• Day count convention: ACT/360	• Day count convention: 30/360 • Payment frequency: yearly
GBP	• Day count convention: ACT/365	• Day count convention: ACT/365 • Payment frequency: half-yearly
USD	• Day count convention: ACT/360	• Day count convention: 30/360 or ACT/360[2] • Payment frequency: yearly
PLN	• Day count convention: ACT/365	• Day count convention: ACT/ACT • Payment frequency: yearly

[2] 30/360 for New York, ACT/360 for London.

6.2.3.3. TBS

A tenor basis swap (TBS) is a contract in which two counterparties agree to exchange (swap) two sets of cash flows based on different market reference rates. TBS can be seen as a contract for exchanging the floating-rate note for some other floating-rate note connected with different market reference rate. In order to minimize the counterparty credit risk, the TBS contract assumes no exchange of notional. The cash flow scheme for TBS contract in PLN is presented on Figure 2.

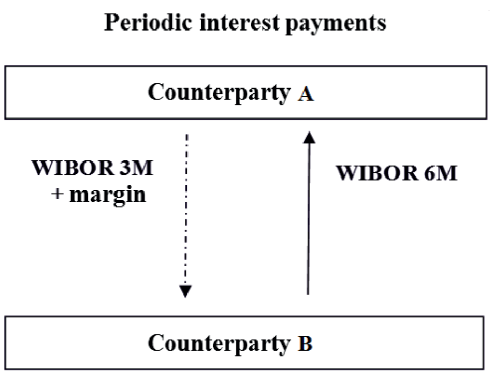

Figure 2. The cash flow scheme for TBS contract in PLN

In market practice, the TBS margin is usually connected with the leg with higher payment frequency. For example, for the TBS contract exchanging WIBOR 3M for WIBOR 6M, the margin will be added to the payments calculated using WIBOR 3M rate.

TBS contract payments are calculated using different market reference rates. As a result, TBS contracts cannot be used as a hedging tool for interest rate risk hedging. However, TBS contracts can serve as a perfect hedging instrument of the basis risk i.e. risk of uncorrelated changes of different market reference rates (Flavell, 2006, p. 137).

TBS fair value at any time t before the maturity date can be calculated using the following formula (assuming one receives margin connected with TBS contract):

$$TBS(t) = N \cdot \sum_{j=1}^{M} \left[\left(T_j - T_{j-1}\right) \cdot \left(F\left(t, T_{j-1}, T_j\right) + m\right) \cdot P\left(t, T_j\right) \right] +$$
$$- N \cdot \sum_{i=1}^{L} \left[\left(T_i - T_{i-1}\right) \cdot F\left(t, T_{i-1}, T_i\right) \cdot P\left(t, T_i\right) \right], \quad (16)$$

where:

N – contract notional,
m – TBS margin,
L – number of payments paid from time t until the maturity date T_L,
M – number of payments received from time t until the maturity date T_M $(T_L = T_M)$.

In practice, when pricing TBS contracts one needs to consider the market conventions that depend on currency of the contract. They are similar to market conventions used in the case of FRA contracts and IRS floating leg for a given currency.

6.2.3.4. OIS

An overnight indexed swap (OIS) is a contract in which two counterparties agree to exchange (swap) two sets of cash flows. An OIS contract assumes one party making payments calculated using the agreed fixed interest rate (fixed leg), while the other makes payments calculated using the agreed overnight reference rate (floating leg). For example, the POLONIA rate is used for PLN currency, whereas the EONIA rate is used in the Eurozone. In order to minimize the counterparty credit risk, the OIS contract assumes no exchange of the notional. In most cases, the maturity of OIS contracts is usually short, not exceeding one year (Flavell, 2006, pp. 131-134). The payments are exchanged on pre-agreed payment dates. The fixed leg payment on date T_i (CF_fixed) is calculated using the following formula:

$$CF\_fixed(OIS) = N \cdot (T_i - T_{i-1}) \cdot K, \tag{17}$$

where:

N – contract notional,
K – pre-agreed contract fixed rate.

The floating leg payment on date T_i $(CF\_floating)$ is calculated using the following formula:

$$CF\_floating(OIS) = N \cdot \left[\prod_{i=1}^{M} \left(1 + L_{ON}(T_{i-1}, T_i) \cdot (T_i - T_{i-1})\right) - 1 \right], \tag{18}$$

where:

N – contract notional,
$L_{ON}(T_{i-1}, T_i)$ – market reference overnight rate for the period (T_{i-1}, T_i),
M – number of business days in a given interest period.

For OIS contracts with maturity below one year, there is only one exchange of payments occurring at contract maturity date.

OIS contracts are used in market practice mainly for hedging or speculation on changes in the overnight money market rate.

OIS fair value at any time *t* before the maturity date can be calculated using the following formula:

$$OIS(t) = PV\_floating(t) - PV\_fixed(t). \tag{19}$$

The fair value of fixed leg at any time *t* before the maturity date can be calculated using the following formula (assuming only one fixed leg payment):

$$PV\_fixed(t) = N \cdot K \cdot (T_M - T_0) \cdot P(t, T_M). \tag{20}$$

The fair value of the floating leg at any time *t* before the maturity date can be calculated using the following formula (assuming only one floating leg payment):

$$PV\_floating(t) = N \cdot \left[\prod_{i=1}^{M} \left(1 + F(t, T_{i-1}, T_i) \cdot (T_i - T_{i-1})\right) - 1 \right] \cdot P(t, T_M). \tag{21}$$

In practice, when pricing OIS contracts one needs to consider the market conventions that depend on the currency of the contract. Table 3 summarizes market conventions for OIS contracts for key currencies and PLN.

Table 3. Market conventions for OIS contracts for key currencies and PLN

Currency	Floating leg	Fixed leg
CHF	• Overnight reference rate: SARON • Day count convention: ACT/360	• Day count convention: ACT/360 • Payment frequency: o one at maturity date (transactions with maturity up to 1 year) o yearly (transactions with maturity over 1 year)
EUR	• Overnight reference rate: EONIA • Day count convention: ACT/360	• Day count convention: ACT/360 • Payment frequency: o one at maturity date (transactions with maturity up to 1 year) o yearly (transactions with maturity over 1 year)
GBP	• Overnight reference rate: SONIA • Day count convention: ACT/365	• Day count convention: ACT/365 • Payment frequency: o one at maturity date (transactions with maturity up to 1 year) o yearly (transactions with maturity over 1 year)

Table 3 – cont.

Currency	Floating leg	Fixed leg
USD	• Overnight reference rate: Fed Fund • Day count convention: ACT/360	• Day count convention: ACT/360 • Payment frequency: ○ one at maturity date (transactions with maturity up to 1 year) ○ yearly (transactions with maturity over 1 year)
PLN	• Overnight reference rate: POLONIA • Day count convention: ACT/365	• Day count convention: ACT/365 • Payment frequency: ○ one at maturity date (transactions with maturity up to 1 year) ○ yearly (transactions with maturity over 1 year)

6.2.4. Cross-currency trades

In this section, we describe the main features of key linear cross-currency derivatives. Instruments described in this section are widely used by market participants for funding and hedging. They also serve as building blocks for bootstrapping algorithms described in the next subchapter.

6.2.4.1. FX swap

FX swaps are mainly used by financial institutions to obtain financing in a foreign currency (Baba, Packer, & Nagano, 2008, p. 75). Under this contract, both counterparties agree to exchange two amounts in two different currencies. The first exchange occurs at the onset of the transaction (spot leg), the other at the maturity date (forward rate). The cash flow scheme for EUR/PLN FX swap contract is presented on Figure 3. We assume that the notional of the transaction is N euro, current EUR/PLN FX spot rate is at S and EUR/PLN forward rate is at F.

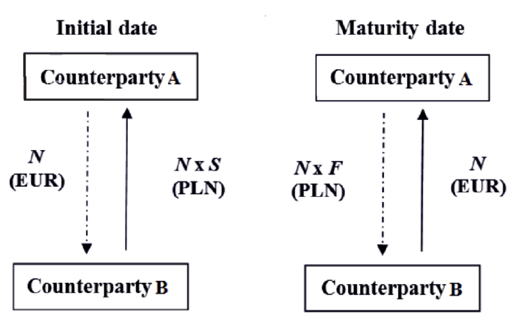

Figure 3. The cash flow scheme for EUR/PLN FX swap contract

The cost of obtaining financing in the foreign currency by the means of FX swap contract equals the difference between current FX spot rate and FX forward rate i.e. $F - S$. The FX forward rate consistent with the market expectations can be calculated with a formula based on interest rate parity (Weron & Weron, 2005, p. 87):

$$F(t,T) = S(t) \cdot \frac{1 + (T-t) \cdot R_d(t,T)}{1 + (T-t) \cdot R_f(t,T)}, \tag{22}$$

where:

$S(t)$ – FX spot rate expressed as one unit of foreign currency expressed in a domestic currency,
$F(t,T)$ – FX forward rate expressed as one unit of foreign currency expressed in a domestic currency,
T – time to maturity (in years),
$R_d(t,T)$ – spot rate for the domestic currency,
$R_f(t,T)$ – spot rate for the foreign currency.

The FX forward rate can also be calculated by using prices of zero-coupon bonds for foreign and domestic currency – see for example (Kenyon & Stamm, 2012, p. 10):

$$F(t,T) = S(t) \cdot \frac{P_f(t,T)}{P_d(t,T)}, \tag{23}$$

where:

$P_d(t,T)$ – price of the domestic currency zero-coupon bond,
$P_f(t,T)$ – price of the foreign currency zero-coupon bond.

The market convention assumes quoting FX swap prices by means of swap points i.e. expressed in basis points difference between the FX forward rate for the transaction and current FX spot rate:

$$F_{mkt}(t,T) = S(t) + Swp(t,T), \tag{24}$$

where:

$F_{mkt}(t,T)$ – FX forward rate for the maturity date T,
$Swp(t,T)$ – swap points quotation for the maturity date T.

Swap points levels for various maturity dates are set by utilizing the interest rate parity relationship for the currencies in question as well as market demand for a given currency pair. The greater the demand, the higher the swap points level and, as a result, the higher the cost of obtaining financing in the foreign currency by means of an FX swap contract.

The fair value of the forward leg at any time *t* before the maturity date can be calculated using the following formula:

$$FX(t,T) = \left[F(t,T) - K\right] \cdot N \cdot P_d(t,T), \tag{25}$$

where:

K – pre-agreed FX rate for the transaction expressed as one unit of foreign currency expressed in a domestic currency,

N – contract notional expressed in a foreign currency.

6.2.4.2. CIRS

A cross currency interest rate swap (CIRS) is a transaction in which two counterparties agree to exchange cash flows in two different currencies. The standard market practice is to exchange two notionals in two different currencies at the onset of the transaction, then pay interest-based payments throughout the life of the transaction and swap back notionals at the maturity date. Therefore, a CIRS transaction can be seen as a combination of two synthetic foreign currency loans provided to each other by participants in a transaction (Flavell, 2006, p. 2).

The key feature of a CIRS transaction is a periodical exchange of interest-based payments in different currencies. Depending on the interest type, there are three main CIRS contract types:

1. Fixed to Fixed Cross Currency Swap: in the specialist literature this contract type is often seen as an equivalent of long-term FX swap transaction – see i.e. (Flavell, 2006, p. 234; Clark, 2011, p. 245-246).
2. Fixed to Floating Cross Currency Swap: this contract type is often used by corporates to hedge FX as well as interest rate risk connected with foreign denominated financing (Flavell, 2006, p. 224).
3. Cross Currency Basis Swap (CCBS): a contract in which two counterparties agree to exchange (swap) two sets of cash flows based on different market reference rates in two different currencies.

The cash flow scheme for EUR/PLN CCBS contract is presented in Figure 4. We assume that the notional of the transaction is N_X euro and N_Y zloty.

CCBS contract payments are calculated using different market reference rates. As a result, CCBS contracts cannot be used as a hedging tool for interest rate risk hedging. However, CCBS contracts are often used by market participants to obtain long-term financing in foreign currencies (Fruchard, Zammouri, & Willems, 1995, p. 70).

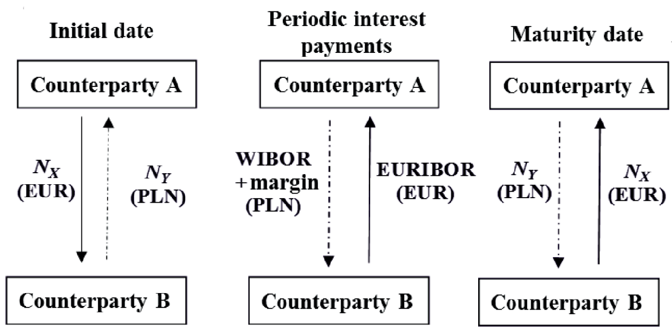

Figure 4. The cash flow scheme for EUR/PLN CCBS contract

CCBS fair value at any time t before the maturity date can be calculated using the following formula (assuming one receives the margin connected with a CCBS contract):

$$CCBS_X(t) = N_X \cdot \left\{ -1 + \sum_{j=1}^{M} \left[\left(T_j - T_{j-1}\right) \cdot \left(F_X\left(t, T_{j-1}, T_j\right) + b\right) \cdot P_X\left(t, T_j\right) \right] + P_X\left(t, T_M\right) \right\} +$$

$$-N_Y \cdot S_{Y/X}(t) \cdot \left\{ -1 + \sum_{i=1}^{L} \left[F_Y\left(t, T_{i-1}, T_i\right) \cdot \left(T_i - T_{i-1}\right) \cdot P_Y\left(t, T_i\right) \right] + P_Y\left(t, T_L\right) \right\}, \quad (26)$$

where:

b	– CCBS margin,
$P_X(t, T_j)$	– price of zero-coupon bond for currency X,
$P_Y(t, T_i)$	– price of zero-coupon bond for currency Y,
$F_X(t, T_{j-1}, T_j)$	– forward rate (simple compounding) for currency X,
$F_Y(t, T_{i-1}, T_i)$	– forward rate (simple compounding) for currency Y,
N_Y	– contract notional for currency Y,
N_X	– contract notional for currency X,
$S_{Y/X}(t)$	– spot FX rate expressed as one unit of currency Y expressed in currency X,
L	– number of payments paid from time t until the maturity date T_L,
M	– number of payments received from time t until the maturity date T_M ($T_L = T_M$).

In practice, when pricing CCBS contracts one needs to consider the market conventions that depend on the currency of the contract. They are similar to market conventions used in the case of FRA contracts and the IRS floating leg for a given currency.

6.3. Zero-coupon and discount curves construction

This subchapter gives a practical introduction to zero-coupon and discount curves construction methods on the basis of the existing market data.

Term structure of interest rates is usually defined as a graph of function mapping maturities into rates. Such a graph is often called the zero-coupon curve (Brigo & Mercurio, 2001, p. 9). Several different curves may be deduced from interest rate market quotes. However, in this subchapter we concentrate on zero-coupon curves created on the basis of market quotes from the interbank market, i.e. money market rates (LIBOR, WIBOR, etc.), as well as derivatives with money market rates as underlying instruments (FRA, IRS, etc.).

Using the relationship described by formula (1), the zero-bond (discount) curve can be created from the zero-coupon curve. The zero-bond discount curve at time t is the graph of the function (Brigo & Mercurio, 2001, p.10):

$$T \to P(t,T),\ T > t. \tag{27}$$

Under the assumption that interest rates are non-negative, the discount curve is a continuous, monotonically decreasing function, where $P(t,T)$ can only take values from the interval $(0,1)$ (Andersen & Piterbarg, 2010, p. 230).

In Figures 5 and 6 we present the discount curves produced accordingly for an upward slopping term-structure of interest rates and a downward slopping term-structure of interest rates.

6.3.1. Bootstrapping the money market rates

The procedure for the zero-coupon curve construction usually includes two steps. In the first step, one needs to select the set of securities (a benchmark set). The choice of securities in the benchmark set depends on the market under consideration. For our purposes, the benchmark set is constructed on the basis of market quotes of the following instruments:

- money market deposits used for the short-end of the curve[3];
- interest rate futures or FRAs used for the middle area of the curve;
- IRS used for the long-end of the curve.

One of the most popular methods used in market practice for zero-coupon and discount curves construction is a procedure known as bootstrapping. Bootstrapping is an iterative procedure that allows one to obtain a discount curve based

[3] Often IBOR rates e.g. WIBOR for PLN, EURIBOR for EUR, LIBOR for USD.

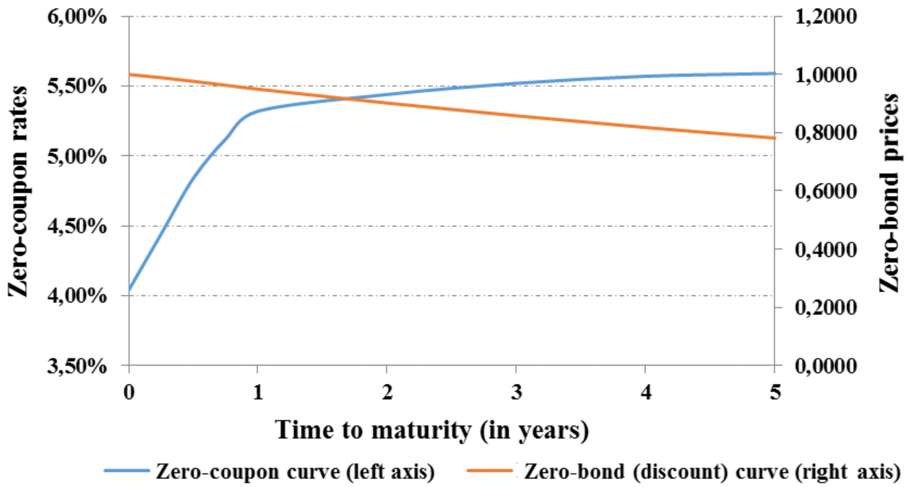

Figure 5. The discount curve for an upward slopping term-structure of interest rates

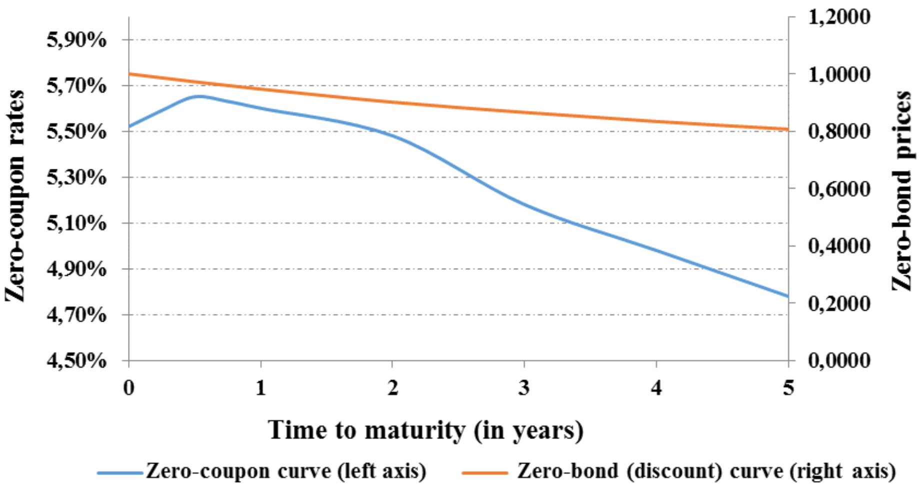

Figure 6. The discount curve for a downward slopping term-structure of interest rates

on corresponding market quotations of liquid instruments. The basic idea can be described by the following algorithm (Andersen & Piterbarg, 2010, p. 234):

1. Let $P(t, t_j)$ be known for $t_j \le T_{i-1}$, such that prices for benchmark securities with the maturity of $T_1, T_2, \ldots, T_{i-1}$ are matched.
2. Make a guess for $P(t, T_i)$.
3. Use an interpolation rule to fill in $P\left(t, t_j\right)$ where $T_{i-1} < t_j < T_i$.

4. Compute $V(t, T_i)$ from the now-known values of $P(t, t_j)$, $t_j \leq T_i$.
5. If $V(t, T_i)$ equals the value observed in the market then stop. Otherwise return to Step 2.
6. If $T_i < T_N$, return to Step 1 and repeat the algorithm for T_{i+1}.

The algorithm is based on the following assumptions:

- The algorithm uses market data available as at t.
- There are N market instruments available with maturities $T_1, T_2, \ldots, T_N$.
- The market instruments used in the algorithm have rising maturities:

$$T_i > T_{i-1} \text{ for } i = 2, 3, \ldots N.$$

- Each of the market instruments used in the algorithm has a market price $V_M(t, T_i)$ available as at t.
- The maturity dates $T_1, T_2, \ldots, T_N$ of the market instruments used in the algorithm constitute so called knot points. Zero-bond prices $P(t, t_j)$ for date between the knot points are calculated using the interpolation algorithm.

The prices of market instruments can be represented by pricing expressions presented in sections 6.2.3 and 6.2.4. Below we present how these equations can be used to obtain the zero-bond prices from the market quotes of various instruments.

a) *Money market rates*

The zero-bond prices for the short end of the curve are usually derived using money market rates. The money market rates are usually quoted for the spot date,[4] assuming simple compounding. The zero-bond price (discount factor) for the maturity date T_i can be calculated from the money market rates using the following formula:

$$P(t, T_i) = \frac{1}{1 + R(t, T_i) \cdot (T_i - t)}, \tag{28}$$

where:

$R(t, T_i)$ – money market rate for the maturity date T_i.

Case 1

Bootstrapping the money market rates using formula (28). We are assuming the ACT/365 day count convention.

[4] The spot date is the date when the transaction is initiated. For most currencies the spot date is usually two business days after the date the transaction is entered into.

Table 4. Bootstrapping the money market rates

Maturity T_i	Rate $R(t, T_i)$ (%)	Days	Application of formula (28)	Discount factor $P(t, T_i)$
ON	1.58	1	= 1/(1 + 1.58% · 1/365)	0.99996
1W	1.60	9	= 1/(1 + 1.60% · 9/365)	0.99961
1M	1.66	33	= 1/(1 + 1.66% · 33/365)	0.99850
3M	1.73	94	= 1/(1 + 1.73% · 94/365)	0.99556
6M	1.81	184	= 1/(1 + 1.81% · 184/365)	0.99096

b) *Forward rates*

Forward rates are usually used to bridge the gap between money market and swap rates and allow to calculate the zero-bond prices for the for the middle area of the curve. Using market quotes of FRAs, the zero-bond price for the maturity date T_i can be calculated from the FRA rates using the following formula[5]:

$$P(t, T_i) = \frac{P(t, T_{i-1})}{1 + K(t, T_{i-1}, T_i) \cdot (T_i - T_{i-1})}, \tag{29}$$

where:

$K(t, T_{i-1}, T_i)$ – FRA rate for the period from T_{i-1} to T_i.

Alternatively, market quotes of interest rate futures can be used to calculate the zero-bond prices for the for the middle area of the curve. Futures contracts are usually used in the bootstrapping process for currencies for which an active market in interest rate futures exists, such as the US dollar interbank market (e.g. 3M eurodollar futures quoted on the Chicago Mercantile Exchange) and EUR (e.g. 3M EURIBOR futures quoted on Eurex).

The forward rate for the period from T_{i-1} to T_i can be calculated from market price of interest rate futures by using the following formula:

$$Q(t, T_{i-1}, T_i) = \frac{100 - FP(t, T_{i-1}, T_i)}{100}, \tag{30}$$

where:

$Q(t, T_{i-1}, T_i)$ – forward rate for the period from T_{i-1} to T_i calculated using market price of interest rate futures,

$FP(t, T_{i-1}, T_i)$ – market price of interest rate futures for period from T_{i-1} to T_i.

[5] We assume that $P(t, T_{i-1})$ is known and $T_{i-1} < T_i$.

Forward rates calculated using the market price of interest rate futures cannot be directly used in the bootstrapping algorithm as they contain the additional component called the convexity adjustment (Hull, 2009, pp. 138-139). A convexity adjustment is directly related to the daily settlement of exchange-traded interest rate futures. In the case of an increase in market interest rates, a long position in interest rate futures generates losses that must be immediately covered by loans obtained at rising interest rates. The reverse is the case when market interest rates fall, resulting in gains from long positions in interest rate futures being reinvested at lower market rates. A long position in interest rate futures gives rise to higher losses in case of interest rate increases and lower returns in case of interest rate falls than a corresponding short position in the FRA contract; therefore, it contains an additional factor i.e. the convexity adjustment.

Using the calculated convexity adjustment,[6] the zero-bond price (discount factor) for the maturity date T_i can be obtained from the forward rate calculated using market prices of interest rate futures with the following formula:

$$P(t,T_i)=\frac{P(t,T_{i-1})}{1+\left[Q(t,T_{i-1},T_i)-CA(t,T_{i-1},T_i)\right]\cdot(T_i-T_{i-1})}, \tag{31}$$

where:

$CA(t,T_{i-1},T_i)$ – convexity adjustment for period from T_{i-1} to T_i.

Case 2

Bootstrapping the money market rates using formula (29). We are assuming the ACT/365 day count convention.

Table 5. Bootstrapping the forward rates

Maturity T_i	Rate $K(t,T_{i-1},T_i)$ (%)	Days	Application of formula (29)[7]	Discount factor $P(t,T_i)$
9M	1.90	91	= 0.99096 / (1 + 1.90% · 91/365)	0.98629
12M	1.95	92	= 0.98629 / (1 + 1.95% · 92/365)	0.98146
15M	2.00	92	= 0.98146 / (1 + 2.00% · 92/365)	0.97654

c) *IRS rates*

The zero-bond prices for the long end of the curve are usually derived using IRS rates. The IRS rate can be considered as par yield because it presents the

[6] The description of possible methods for calculation of convexity adjustment can be found in Hull (2009), Flesaker (1993), Kirikos and Novak (1997).

[7] Please note that $P(t,T_{i-1})$ value for the first tenor (9M) i.e. 0.99096 was obtained from 6M tenor from Table 4.

fair value of a fixed rate bond against a floating rate bond which is always worth par at inception.[8] The zero-bond price (discount factor) for the maturity date T_n can therefore be calculated from the IRS rates using the following formula:

$$P(t,T_n)=\frac{1-K\cdot\sum_{i=1}^{n-1}\left[(T_i-T_{i-1})\cdot P(t,T_i)\right]}{1+K\cdot(T_n-T_{n-1})}. \qquad (32)$$

Case 3

Bootstrapping the money market rates using formula (32). We are assuming the ACT/365 day count convention.

Table 6. Bootstrapping the IRS rates

Maturity T_n	Rate K (%)	Days	Application of formula (30)[9]	Discount factor $P(t,T_n)$
2Y	2.25	365	= (1 – 2.25% · 365/365 · 0.98146) / (1 + 2.25% · 365/365)	0.97572
3Y	2.75	365	= (1 – 2.75% · [365/365 · 0.98146 + 365/365 · 0.97572]) / (1 + 2.75% · 365/365)	0.97302

6.3.2. Interpolation

The bootstrapping algorithm presented in 6.3.1 allows us to obtain the discount factor for knot points corresponding to the maturity of the used set of market instruments. In order to obtain discount factors for maturities other than knot points, an appropriate interpolation algorithm should be used. In this section we briefly present the properties of key interpolation algorithms used in the market practice. A more comprehensive analysis of the properties of interest rate curve interpolation algorithms is presented in Hagan and West (2006).

The interpolation algorithms used in market practice and described in the literature are usually divided into three types:

- simple interpolation methods (linear interpolation of zero-coupon rates or zero-coupon bond prices),
- cubic splines,
- forward splines.

To simplify the notation, in this section, it is assumed that $R(t,T)=R(T)$, $P(t,T)=P(T)$ and $f(t,T)=f(T)$.

[8] If no basis spread exists on the market.

[9] The 0.98146 discount factor for 1Y tenor is obtained from Case 2.

6.3.2.1. Simple interpolation methods

Simple interpolation methods allow us to estimate the zero-coupon rate $R(T)$ using only the known values of zero-coupon rates $R(T_i)$ and $R(T_{i+1})$ for two nearest knot points T_i and T_{i+1} where $T_i < T < T_{i+1}$.

One of the most popular methods used in market practice is the raw interpolation (linear on the logarithm of discount factors). This method is very stable and easy to implement. It is usually a base method one implements in a system before any others. One can often find mistakes in more sophisticated algorithms by comparing the raw method with the more sophisticated interpolation method (Hagan & West, 2006, p. 95).

The interpolation algorithm in raw interpolation can be described with the following formula:

$$\ln P(T) = \frac{T - T_i}{T_{i+1} - T_i} \cdot \ln P(T_{i+1}) + \frac{T_{i+1} - T}{T_{i+1} - T_i} \cdot \ln P(T_i), \quad \text{for} \quad T_i < T < T_{i+1}, \quad (33)$$

where:

$P(T)$ – zero-coupon bond price for maturity T.

Using:

$$P(T) = e^{-R(T) \cdot T}, \quad (34)$$

the interpolation algorithm for zero-coupon rates can be derived:

$$R(T) = \frac{T - T_i}{T_{i+1} - T_i} \cdot \frac{T_{i+1}}{T} \cdot R(T_{i+1}) + \frac{T_{i+1} - T}{T_{i+1} - T_i} \cdot \frac{T_i}{T} \cdot R(T_i), \quad \text{for} \quad T_i < T < T_{i+1}. \quad (35)$$

Formula (36) presents the interpolation algorithm for the instantaneous forward rate[10]:

$$f(T) = \frac{T_{i+1}}{T_{i+1} - T_i} \cdot R(T_{i+1}) - \frac{T_i}{T_{i+1} - T_i} \cdot R(T_i), \quad \text{for} \quad T_i < T < T_{i+1}. \quad (36)$$

Function $f(T)$ is not continuous. The instantaneous forward rate value for maturity date T, where $T_i < T < T_{i+1}$, depends only on the values of the two nearest knot points values. Analyzing formula (36) one can also spot that the

[10] The forward rate for the infinitesimal period of time i.e. $f(t, T) = \lim_{S \to T^+} F(t, T, S) = \frac{-\partial P(t, T)}{\partial T}$.

raw interpolation algorithm produces constant forward rates at each interval defined by the knot points.

Figure 7 presents the zero-coupon rates curve as well as forward rates curve for time to maturities interval from 1 to 5 years estimated using the raw interpolation algorithm based on sample data from the Polish market as at 28th February 2011.

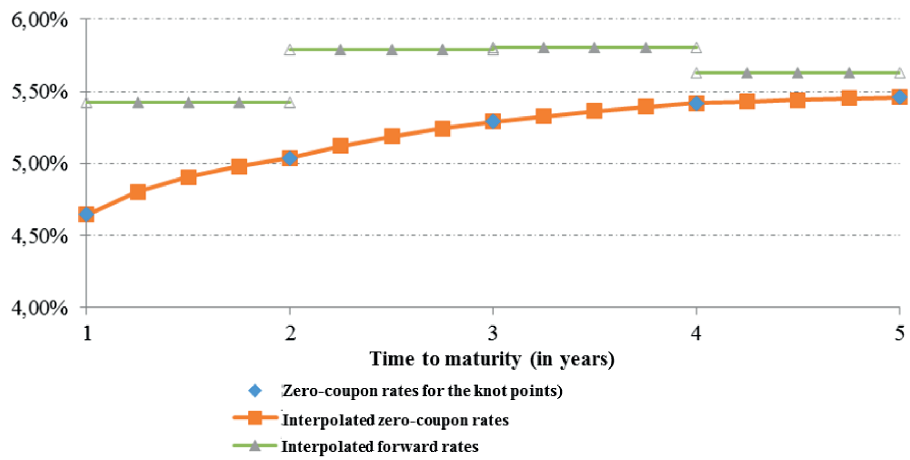

Figure 7. The zero-coupon rates and the forward rates estimated using the raw interpolation algorithm

Source: Own calculation based on sample data.

6.3.2.2. Cubic splines

The interpolation algorithm in cubic spline interpolation can be described with the following formula:

$$R(T)=a_i+b_i\cdot(T-T_i)+c_i\cdot(T-T_i)^2+d_i\cdot(T-T_i)^3,\ \ T_i\le T\le T_{i+1}. \quad (37)$$

In order to calibrate the set of parameters (a_i, b_i, c_i, d_i) (allowing us to estimate the zero-coupon rate for any maturity date T, where $T_1 < T < T_n$, the following constraints need to be observed (Hagan & West, 2006, p. 97):

- the interpolating function meets the zero-coupon rates at knot points:

$$a_i = R(T_i), \quad (38)$$

for each $i=1, 2, \dots, n-1,$ and for $i=n$ the following condition is meet:

$$a_{n-1}+b_{n-1}\cdot(T_n-T_{n-1})+c_{n-1}\cdot(T_n-T_{n-1})^2+d_{n-1}\cdot(T_n-T_{n-1})^3=a_n=R(T_n), \quad (39)$$

- the interpolating function is continuous:

$$a_i + b_i \cdot (T_{i+1} - T_i) + c_i \cdot (T_{i+1} - T_i)^2 + d_i \cdot (T_{i+1} - T_i)^3 = a_{i+1}, \tag{40}$$

for each $i = 1, 2, \ldots, n-2$;

- the interpolating function is differentiable:

$$b_i + 2c_i \cdot (T_{i+1} - T_i) + 3d_i \cdot (T_{i+1} - T_i)^2 = b_{i+1}. \tag{41}$$

for each $i = 1, 2, \ldots, n-2$.

Formula (42) presents the interpolation algorithm for the instantaneous forward rate (Hagan & West, 2006, p. 97):

$$f(T) = a_i + b_i \cdot (2T - T_i) + c_i \cdot (T - T_i) \cdot (3T - T_i) + \\ + d_i \cdot (T - T_i)^2 \cdot (4T - T_i), \; T_i \le T \le T_{i+1}. \tag{42}$$

The conditions defined by (38)-(41) create a system of $3n-4$ equations with $4n-4$ unknowns. Thus, n linear constraints still need to be specified.

One of the most popular cubic splines used in market practice is the natural cubic spline. For natural cubic spline the following n additional conditions are specified as follows:

- the interpolating function is twice differentiable:

$$c_i + 3d_i \cdot (T_{i+1} - T_i) = c_{i+1}, \tag{43}$$

for each $i = 1, 2, \ldots, n-2$;

- the second derivative at each endpoint (i.e. $i = 1$, $i = n$) is zero:

$$R''(T_1) = R''(T_n) = 0. \tag{44}$$

Additional n conditions allow us to create a system of $4n-4$ equations with $4n-4$ unknowns. The system can be solved using the algorithm as described by Burden and Faires (2011, p. 149-150):

- **Step 1:** For $i = 1, 2, \ldots, n-1$ calculate:

$$h_i = T_{i+1} - T_i;$$

$$a_i = R(T_i);$$

- **Step 2:** For $i=2, \ldots, n-1$ calculate:

$$\alpha_i = \frac{3}{h_i} \cdot (a_{i+1} - a_i) - \frac{3}{h_{i-1}} \cdot (a_i - a_{i-1});$$

- **Step 3:** For $i=1$ calculate:

$$l_1 = 1;$$
$$u_1 = 0;$$
$$z_1 = 0;$$

- **Step 4:** For $i=2, \ldots, n-1$ calculate:

$$l_i = 2 \cdot (T_{i+1} - T_{i-1}) - h_{i-1} \cdot u_{i-1};$$
$$u_i = \frac{h_i}{l_i};$$
$$z_i = \frac{\alpha_i - h_{i-1} \cdot z_{i-1}}{l_i};$$

- **Step 5:** For $i=n$ calculate:

$$l_n = 1;$$
$$u_n = 0;$$
$$z_n = 0;$$

- **Step 6:** For $j=n-1, n-2, \ldots, 1$ calculate retrospectively[11]:

$$c_j = z_j - u_j \cdot c_{j+1};$$
$$b_j = \frac{a_{j+1} - a_j}{h_j} - h_j \frac{c_{j+1} + 2c_j}{3};$$
$$d_j = \frac{c_{j+1} - c_j}{3h_j}.$$

[11] Please note that (44) implies that $c_1 = c_n = 0$.

Figure 8 presents the zero-coupon rates curve as well as forward rates curve for time to maturities interval from 1 to 5 years estimated using the natural cubic spline algorithm based on sample data from the Polish market as at 28th February 2011.

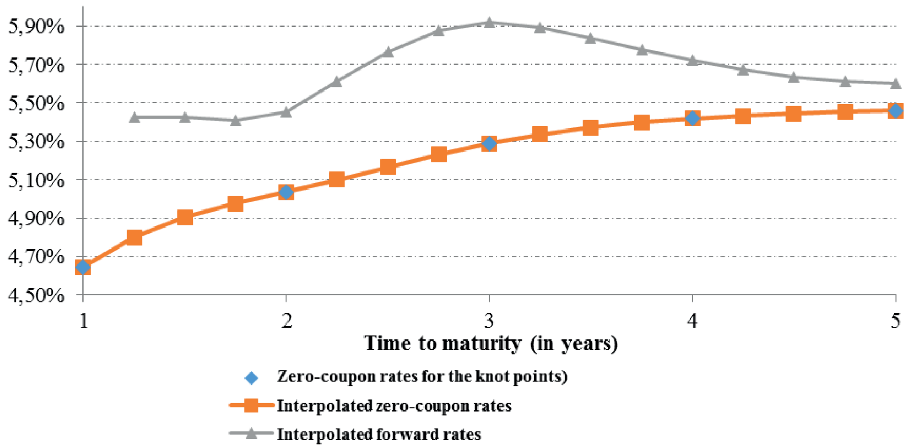

Figure 8. The zero-coupon rates and the forward rates estimated using the natural cubic spline algorithm

Source: Own calculation based on sample data.

The method allows us to estimate the continuous zero-coupon rates curves as well as the forward rates curves. However, for curves with a more sparse set of nodes, the resulting curve is too convex ('bulging') between points which are a fair distance away (Hagan & West, 2006, p. 97).

6.4. Bootstrapping of zero curves in the multi-curve framework

In this subchapter, we present the rationale for the pricing framework that emerged as a direct result of the 2007-2009 financial crisis.

6.4.1. What has changed and why?

The credit and liquidity crisis that started in August 2007 created a phenomenon that had not been taken into account before (mainly because of its negligible impact): there was a strong increase in the basis spreads quoted on the market between floating rate instruments of different tenors or currencies (e.g. WIBOR 3M vs EURIBOR 3M, WIBOR 3M vs WIBOR 6M). It was the result of the increased liquidity and credit risk and the corresponding preference of market participants for instruments with higher payment frequency – see (Taylor & Williams, 2009; Schwarz, 2010; Soultanaeva & Strömqvist, 2009; Michaud & Upper, 2008). There were also other indicators such as the divergence between interbank unsecured deposit rates (e.g. WIBOR) and overnight based (OIS) rates with the same maturity, or between FRA contacts and the corresponding forward rates implied by consecutive interbank deposits.

Figure 9 presents the evolution of the risk premiums[12] for PLN, EUR and USD for the period from 3rd January 2005 to 30th September 2011. Risk premiums were calculated on the basis of WIBOR, EURIBOR, LIBOR USD as well as OIS quotes for maturities of 3 months.

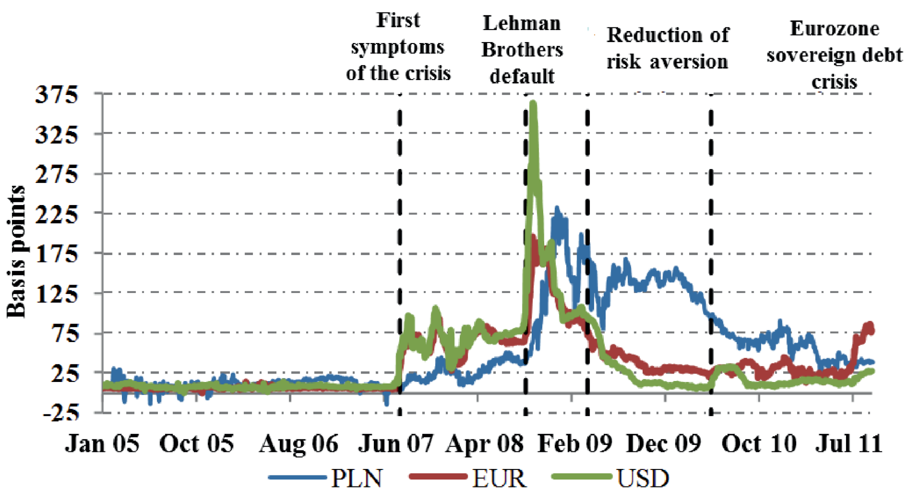

Figure 9. The evolution of the risk premiums for PLN, EUR and USD for the period 3.01.2005-30.09.2011

Figure 10 presents the divergence between FRA contract rates and the corresponding forward rates implied by consecutive interbank deposits. The chart is based on market quotes of FRA 3 · 6 and interbank deposit quotes for 3 and 6 months.

[12] The interbank risk premium is defined in the literature as the spread between the interbank reference rates (e.g. LIBOR, WIBOR, EURIBOR) and OIS transaction rates for the corresponding maturities – see (Thornton, 2009; Kliber & Płuciennik, 2011; Schwarz, 2010).

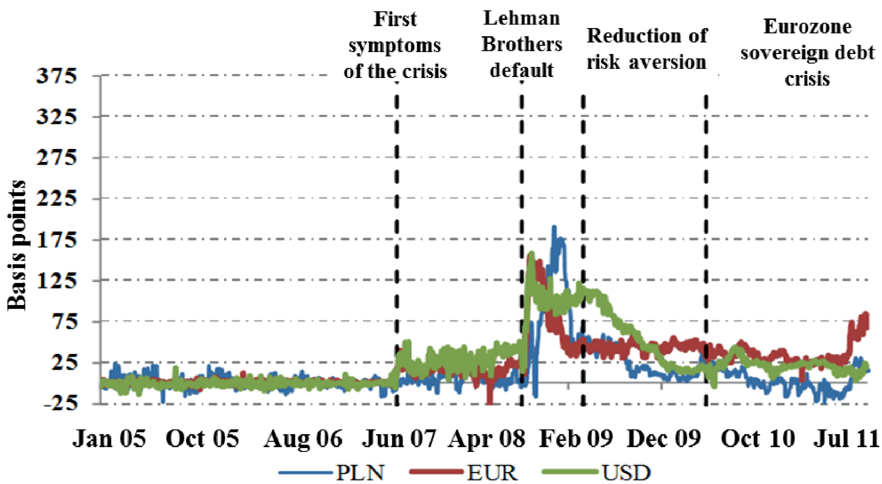

Figure 10. The divergence between FRA 3 · 6 contract rate and the corresponding forward rate implied by interbank deposits for 3 and 6 months for PLN, EUR and USD for the period 3.01.2005-30.09.2011

Source: Own calculation.

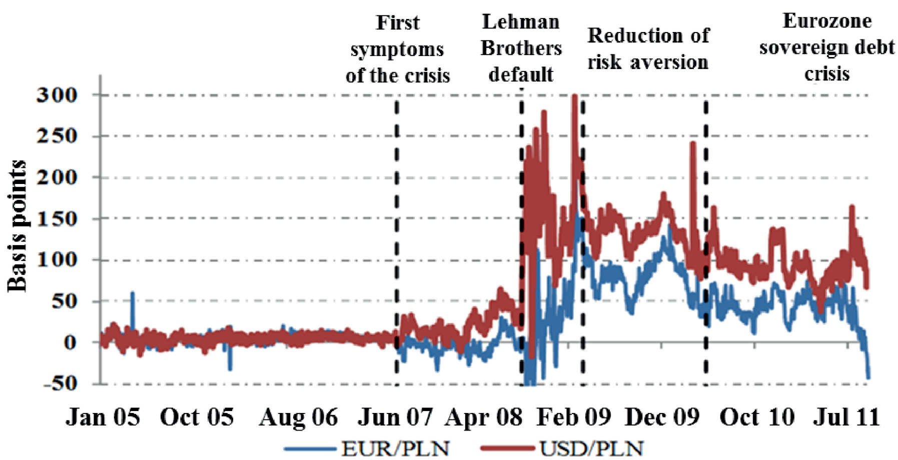

Figure 11. The evolution of currency basis spread for EUR/PLN and USD/PLN currency pairs for the period 3.01.2005-30.09.2011

Figure 11 presents the evolution of currency basis spread for EUR/PLN and USD/PLN currency pairs for the period from 3rd January 2005 to 30th September 2011. Currency basis spreads were calculated one the basis of quotes of 3-month maturity instruments.

As can be seen, it is clear that we cannot treat all interbank deposits rates (as well as quotes of their derivatives) equally as discounting rates in order to price the financial products consistently with the existing market prices. The classical arbitrage arguments that serve as a foundation for the bootstrapping algorithms are no longer consistent with the market. Furthermore, more and more financial contracts include collateral agreements. All this implies that the traditional approach to discount curves construction outlined in 6.3.1 no longer can be applied. The traditional approach does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with different underlying rate tenors and different collateral arrangements (collateral type, collateral currency etc.) nor does it take into account market information carried by basis swaps between different tenors and currencies.

In order to cope with the difficulties described above, the market practice evolved into the procedure that can be outlined below:

1. Build one discounting curve using the preferred procedure.
2. Build multiple discount curves for different currencies taking into account their relationship implied by market quotes of cross currency basis swap instruments.
3. Build multiple forward curves for different tenors (e.g. 1M, 3M, 6M, 12M) taking into account their relationship implied by market quotes of tenor basis swap instruments.
4. Compute on each forward curve the forward rates and the corresponding cash flows relevant for pricing derivatives on the same underlying tenor.
5. Compute the corresponding discount factors using the discounting curve and work out prices by summing up the discounted cash flows.

In the remaining parts of this subchapter, we will outline the details of bootstrapping algorithms used in the new multi-curve framework.

6.4.2. Building the discount curve

In this section, we will describe the algorithms for construction of discount curves under the existence of collateral agreements in the new multi-curve framework.

Under a collateral agreement, the counterparty receives the collateral from the other side of the transaction when the present value of the contract is positive, and needs to pay the margin called the”collateral rate” on the outstanding collateral to the payer. The most commonly used collateral is a currency of developed countries, such as USD, EUR or JPY, and the mark-to-market of the contracts is to be made quite frequently i.e. daily. In the case of cash collateral, the overnight rate for the collateral currency, such as the EONIA rate for EUR, is usually used as the collateral rate (Fujii, Shimada, & Takahashi, 2010, p. 8).

The existence of collateral not only mitigates the counterparty credit risk but also changes the funding cost significantly and hence affects the discount curve construction methodologies. The arguments presented by Fujii and others (2010) as well as Piterbarg (2012) show that the classical approach, as presented in 6.3.1, is no longer appropriate for the pricing of collateralized trades. The cash flows of the collateralized trade should be discounted using the collateral rate. It is therefore critical to determine the future levels of collateral rate for the pricing of collateralized derivatives.

Market instruments whose quotes depend on the expected future overnight rates are called OIS transactions. Based on (20) and (21) as well as taking into account that[13]:

$$F^D\left(t,T_{n-1},T_n\right)=\frac{1}{T_n-T_{n-1}}\cdot\left(\frac{P^D\left(t,T_{n-1}\right)}{P^D\left(t,T_n\right)}-1\right), \tag{45}$$

the value of an OIS transaction at time t can be calculated using the following formula[14]:

$$OIS(t)=1-P^D\left(t,T_N\right)-K\cdot\sum_{m=1}^{M}\left[\left(T_m-T_{m-1}\right)\cdot P^D\left(t,T_m\right)\right], \tag{46}$$

where:

K – pre-agreed contract fixed rate,

M – number of fixed rate payments from t until the maturity date T_M ($T_M = T_N$).

Based on (46) the zero-bond price (discount factor) for currency X, assuming the transaction is cash-collateralized in the same currency, can be calculated from the OIS rates using the following formula:

[13] Please note that for the rest of this chapter we use superscript D to denote market data and results obtained for discount curve. Superscripts j, i, etc. are used for the purposed of forward curves construction i.e. $P_X^D\left(t,T_M\right)$ stands for zero-bond price based in discount curve in currency X whereas $P_X^j\left(t,T_M\right)$ stands for the zero-bond price based on forward curve for reference rate j in currency X.

[14] Please note that formula (21) can be simplified to $1-P\left(t,T_M\right)$. For example for $M = 2$ and $N = 1$ we obtain the following expression:

$$\left[\left(1+F\left(t,T_0,T_1\right)\cdot\left(T_1-T_0\right)\right)\cdot\left(1+F\left(t,T_1,T_2\right)\cdot\left(T_2-T_1\right)\right)-1\right]\cdot P\left(t,T_2\right)=$$

$$=\left[\left(1+\frac{1}{T_1-T_0}\cdot\left(\frac{P\left(t,T_0\right)}{P\left(t,T_1\right)}-1\right)\cdot\left(T_1-T_0\right)\right)\cdot\left(1+\frac{1}{T_2-T_1}\cdot\left(\frac{P\left(t,T_1\right)}{P\left(t,T_2\right)}-1\right)\cdot\left(T_2-T_1\right)\right)\right]\cdot P\left(t,T_2\right)=$$

$$=\left[\frac{P\left(t,T_0\right)}{P\left(t,T_1\right)}\cdot\frac{P\left(t,T_1\right)}{P\left(t,T_2\right)}-1\right]\cdot P\left(t,T_2\right)=1-P\left(t,T_2\right).$$

$$P_X^D\left(t, T_M\right)=\frac{1-K_X^D\left(t, T_M\right) \cdot \sum_{m=1}^{M-1}\left[\left(T_m-T_{m-1}\right) \cdot P_X^D\left(t, T_m\right)\right]}{1+K_X^D\left(t, T_M\right) \cdot\left[T_M-T_{M-1}\right]}, \qquad (47)$$

where:

$K_X^D\left(t, T_M\right)$ – pre-agreed contract fixed rate for the maturity date T_M.

The presence of the risk premium results in varying levels of the discount curves used for pricing collateralized and uncollateralized transactions.

Case 4

Building the discount curve for collateralized transactions (collateral currency equals deal currency) by utilizing OIS rates and using formula (47).

Table 7. Bootstrapping the OIS rates

Maturity T_M	OIS rate $K_X^D(t, T_M)$ (%)	Days	Discount factor $P_X^D(t, T_M)$	Sum in numerator[15]	Application of formula (47)
3M	2.00	90	0.9951	0.2454	= (1 – 2.00% · 0.00) / (1 + 2.00% · 90/365)
6M	2.25	90	0.9890	0.4892	= (1 – 2.25% · 0.2454) / (1 + 2.25% · 90/365)
9M	2.50	90	0.9817	0.7313	= (1 – 2.50% · 0.4892) / (1 + 2.25% · 90/365)
12M	2.75	90	0.9733	0.9713	= (1 – 2.50% · 0.7313) / (1 + 2.50% · 90/365)

The presence of the risk premium results in higher discount rates used for pricing of uncollateralized transactions. As a result, the fair value of two identical transactions, one of which is collateralized and the other which is not collateralized, will differ. A rational investor expects higher return from uncollateralized transaction to compensate him/her for the higher counterparty credit risk connected with the transaction, which implies higher discount rates used to price this type of transaction.

In reality, the collateral currency is often different than the currency of the transaction. The difference between the transaction currency and the collateral currency affects the cash flows associated with the transaction. In situations

[15] Sum in numerator stands for $\sum_{m=1}^{M-1}\left[\left(T_m-T_{m-1}\right) \cdot P_X^D\left(t, T_m\right)\right]$ part in formula (47). For example 0.4892 = 0.9890 · 90/365 + 0.9951 · 90/365.

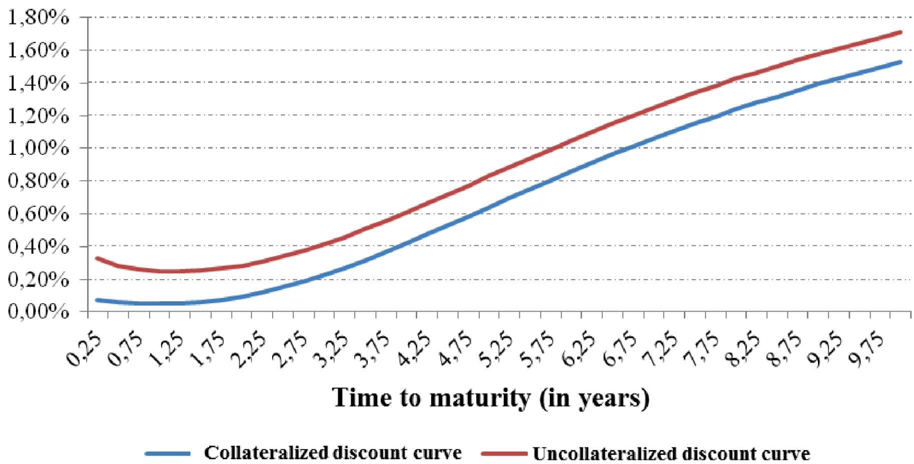

Figure 12. Sample discount curves used for pricing of collateralized and uncollateralized transactions in EUR currency
Source: Own calculation based on sample data.

where the currency of the transaction (*X*) is different than the collateral currency (*Y*), the total cash flows of the IRS transaction can be described by the following formula:

$$CF_{IRS} = \sum_{i=1}^{M} C_i - \sum_{j=1}^{N} L_j + \sum_{l=1}^{K-1}\left[S_{\frac{Y}{X}}(t_{l+1}) \cdot V_Y(t_l)\right] - $$
$$+ \sum_{l=2}^{K}\left[S_{Y/X}(t_l) \cdot V_Y(t_{l-1}) \cdot \left(1 + c \cdot (t_l - t_{l-1})\right)\right] \tag{48}$$

where:

$\sum_{i=1}^{M} C_i$ – sum of cash flows from the fixed leg of the IRS transaction,

$\sum_{j=1}^{N} L_j$ – sum of cash flows from the floating leg of the IRS transaction,

$\sum_{l=1}^{K-1}\left[S_{Y/X}(t_{l+1}) \cdot V_Y(t_l)\right] - \sum_{l=2}^{K}\left[S_{Y/X}(t_l) \cdot V_Y(t_{l-1}) \cdot \left(1 + c \cdot (t_l - t_{l-1})\right)\right]$ – sum of cash flows from the collateral exchange,

c – collateral rate,

$S_{Y/X}(t)$ – spot FX rate expressed as one unit of currency *Y* expressed in currency *X*.

The difference in the collateral currency and the transaction currency creates an additional market risk and liquidity risk associated with the transaction. Fujii and others (2010) and McCloud (2013) show that cash flows from collateralized transactions, where the collateral currency is different than the transaction currency, should be discounted using discount rates that take into account the cost of hedging of FX risk connected with the transaction.

Below we presented how the market quotes of FX swaps and cross currency basis swaps (CCBS) can be incorporated into the bootstrapping algorithm to obtain the zero-bond prices appropriate to price the collateralized transactions where the collateral currency is different than the currency of the transaction.

a) *FX Swaps*

The zero-bond prices for the short end of the curve can be derived using FX swaps quotes. The zero-bond price (discount factor) for the maturity date T_i can be calculated from the money market rates using the following formula (assuming the zero-bond price $P_X^D(t,T_i)$ for the currency X is known):

$$P_Y^D(t,T_i)=\frac{F_{Y/X}(t,T_i)}{S_{Y/X}(t)}\cdot P_X^D(t,T_i), \tag{49}$$

where:

$F_{Y/X}(t,T_i)$ – forward FX rate for maturity date T_i expressed as one unit of currency Y expressed in currency X.

Case 5

Building the discount curve for collateralized transactions (collateral currency differs from deal currency) by utilizing FX Swaps and using formula (49). As discount factors for currency X we will use discount factors obtained in Case 4.

Table 8. Discount factors from FX Swaps prices

Maturity T_i	Spot FX rate $S_{Y/X}(t)$	Forward FX Rate $F_{Y/X}(t,T_i)$	Discount factor for the currency X $P_X^D(t,T_m)$	Discount factor for the currency Y $P_Y^D(t,T_m)$	Application of formula (49)
3M	4.0000	4.0025	0.9951	0.9957	= 4.0025 / 4.0000 · 0.9890
6M	4.0000	4.0049	0.9890	0.9902	= 4.0049 / 4.0000 · 0.9952

b) *Cross currency basis swaps*

For longer maturities, CCBS quotes are used. Under the assumption that we know the discount curve for currency X, the zero-bond price (discount factor) for currency Y for the maturity date T_i can be calculated from the money market rates using the following formula:

$$P_Y^D(t,T_M)=\frac{K_X^j(t,T_M)\cdot\sum_{m=1}^{M}\left[(T_m-T_{m-1})\cdot P_X^D(t,T_m)\right]}{1+K_Y^j(t,T_M)\cdot[T_M-T_{M-1}]}+$$
$$+\frac{b_{X,Y}^j(t,T_M)\cdot\sum_{m=1}^{M}\left[(T_m-T_{m-1})\cdot P_X^D(t,T_m)\right]+P_X^D(t,T_M)}{1+K_Y^j(t,T_M)\cdot[T_M-T_{M-1}]}+ \qquad (50)$$
$$-\frac{K_Y^j(t,T_M)\cdot\sum_{m=1}^{M-1}\left[(T_m-T_{m-1})\cdot P_Y^D(t,T_m)\right]}{1+K_Y^j(t,T_M)\cdot[T_M-T_{M-1}]},$$

where:

$K_X^j(t,T_M)$ – fixed rate for IRS transaction in currency X based on reference rate j,

$K_Y^j(t,T_M)$ – fixed rate for IRS transaction in currency Y based on reference rate j,

$b_{X,Y}^j(t,T_M)$ – cross currency basis swap margin where margin and reference rate j in currency X is exchanged for reference rate Y.

The formula (50) can be easily derived by using pricing expressions as shown in sections 6.2.3 and 6.2.4 and by applying the non-arbitrage argument for the pricing of portfolio containing receiver IRS in currency X, cross currency basis swap receiving cash flows in currency X and paying cash flows in currency Y and payer IRS in currency Y[16]. This approach is similar to one outlined in detail in Fujii and others (2010).

For example, in order to calculate the discount factors that will be used to price PLN IRS transaction that is collateralized in EUR, we need to calculate the discount curve for EUR currency based on EUR OIS market quotes, then we calculate the discount curve for PLN currency that will be based on the abovementioned EUR discount curve as well as EUR/PLN FX swap and CCBS quotes.

Case 6

Building the discount curve for collateralized transactions (collateral currency differs from deal currency) by utilizing cross-currency basis swaps quotes and using formula (50). As discount factors for currency X we will use discount fac-

[16] The non-arbitrage argument states that the portfolio value at the onset of the transaction should be zero.

Table 9. Discount factors from cross-currency basis swaps quotes [A]

Maturity T_M	IRS rate for the currency X K_X^j (%)	IRS rate for the currency Y $K_Y^j(t,T_M)$ (%)	Days	Discount factor for the currency X $P_X^D(t,T_m)$	Sum in numerator[17]	A $\frac{K_X^j(t,T_M)\cdot\sum_{m=1}^{M}\left[(T_m-T_{m-1})\cdot P_X^D(t,T_m)\right]}{1+K_Y^j(t,T_M)\cdot[T_M-T_{M-1}]}$
9M	2.60	2.35	90	0.9817	0.7313	0.0189 = 2.60% · 0.7313 / (1 + 2.35% · 90/365)
12M	2.80	2.55	90	0.9733	0.9713	0.0270 = 2.80% · 0.9713 / (1 + 2.55% · 90/365)

Table 10. Discount factors from cross-currency basis swaps quotes [B]

Maturity T_M	Cross-currency basis swap margin $b_{X,Y}^j(t,T_M)$ (%)	IRS rate for the currency Y $K_Y^j(t,T_M)$ (%)	Days	Discount factor for the currency X $P_X^D(t,T_m)$	Sum in numerator	B $\frac{b_{X,Y}^j(t,T_M)\cdot\sum_{m=1}^{M}\left[(T_m-T_{m-1})\cdot P_X^D(t,T_m)\right]+P_X^D(t,T_M)}{1+K_Y^j(t,T_M)\cdot[T_M-T_{M-1}]}$
9M	0.10	2.35	90	0.9817	0.7313	0.9768 = (0.10% · 0.7313 + 0.9817) / (1 + 2.35% · 90/365)
12M	0.20	2.55	90	0.9733	0.9713	0.9691 = (0.20% · 0.9713 + 0.9733) / (1 + 2.55% · 90/365)

[17] Sum in numerator stands for $\sum_{m=1}^{M-1}\left[(T_m-T_{m-1})\cdot P_X^D(t,T_m)\right]$ part in formula (50). For example 0.7313 = 0.9890 · 90/365 + 0.9951 · 90/365 + + 0.9817 · 90/365.

Table 11. Discount factors from cross-currency basis swaps quotes [C]

Maturity T_M	IRS rate for the currency Y $K_Y^j(t, T_M)$ (%)	Days	Sum in numerator[18]	C $\frac{K_Y^j(t,T_M)\cdot\sum_{m=1}^{M-1}\left[(T_m - T_{m-1})\cdot P_Y^D(t,T_m)\right]}{1+K_Y^j(t,T_M)\cdot\left[T_M - T_{M-1}\right]}$
9M	2.35	90	0.4897	0.0114 = 2.35% · 0.4897 / (1 + 2.35% · 90/365)
12M	2.55	90	0.7324	0.0186 = 2.55% · 0.7324 / (1 + 2.55% · 90/365)

Table 12. Discount factors from cross-currency basis swaps quotes

Maturity T_M	A	B	B	Discount factor for the currency Y $P_Y^D(t, T_m)$ [A + B – C]
9M	0.0189	0.9768	0.0114	0.9843
12M	0.0270	0.9691	0.0186	0.9776

[18] Sum in numerator stands for $K_X^j(t,T_M)\cdot\sum_{m=1}^{M-1}\left[(T_m - T_{m-1})\cdot P_X^D(t,T_m)\right]$ part in formula (50). For example 0.7324 = 0.9957 · 90/365 + 0.9902 × × 90/365 + 0.9843 · 90/365. Discount factors for currency Y for tenors 3M and 6M are obtained from Case 5.

tors obtained in Case 4. As discount factors for currency Y for shorter tenors (3M and 6M) we will uses discount factors obtained in Case 5.

The differences in risk premiums for different currencies imply different levels of discount rates used for pricing of a transaction in the same currency but with different currencies of the collateral. Figure 13 presents sample discount curves for PLN under the assumption of EUR- and USD-denominated cash collateral.

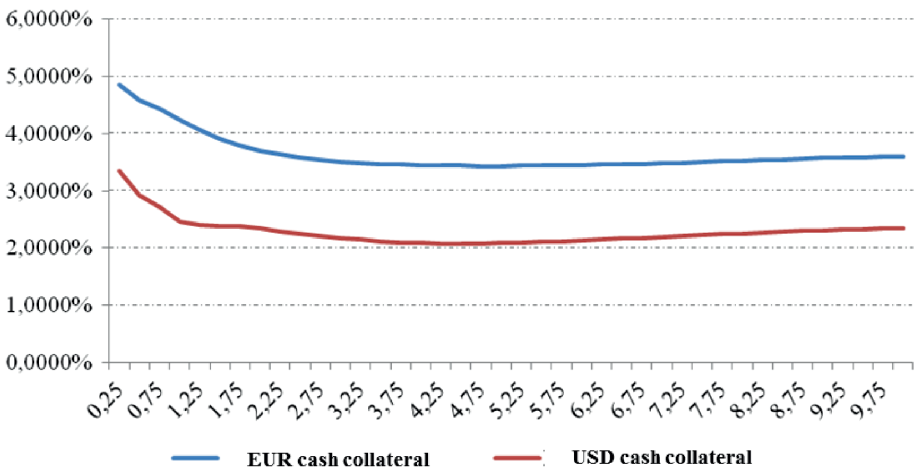

Figure 13. Sample discount curves used for pricing of collateralized transactions in PLN currency under the EUR and USD cash collateral assumption

Source: Own calculation based on sample data.

In case of non-zero CCBS margins, the difference $s^D_{X,Y}(t,T)$ between the discount rates under different collateral assumptions can be approximated using the following formula:

$$s^D_{X,Y}(t,T) = L^D_X(t,T) - L^D_Y(t,T) \cong K^j_Y(t,T) - K^j_X(t,T) - b^j_{X,Y}(t,T). \quad (51)$$

The difference depends on the interest rate parity of the currencies of the collateral modified by the cost of hedging of FX risk connected with the transaction approximated by the cross currency basis swaps margin.

Case 7

Calculating the difference between discount rates for different currencies. As discount factors for currency X we will use discount factors obtained in Case 4. As discount factors for currency Y we will use discount factors obtained in Case 6.

Table 13. Difference between discount rate for currency *X* and *Y*

Maturity T_M	Discount factor for the currency *X* $P_X^D(t,T)$	Discount factor for the currency *Y* $P_Y^D(t,T)$	Discount rate for the currency *X*[19] $L_X^D(t,T)$ (%)	Discount rate for the currency *Y* $L_Y^D(t,T)$ (%)	Difference between discount rate for currency *X* and *Y* (%)
9M	0.9817	0.9843	2.50	2.15	–0.35
12M	0.9733	0.9776	2.75	2.30	–0.45

Table 14. Implied difference between discount rate for currency *X* and *Y*

Maturity T_M	IRS rate for the currency *X* $K_X^j(t,T)$ (%)	IRS rate for the currency *Y* $K_Y^j(t,T)$ (%)	Cross currency basis swap margin $b_{X,Y}^j(t,T)$ (%)	Implied difference between discount rates $s_{X,Y}^D(t,T) \cong K_Y^j(t,T) - +K_X^j(t,T) - b_{X,Y}^j(t,T)$ (%)
9M	2.60	2.35	0.10	–0.35
12M	2.80	2.55	0.20	–0.45

Exercise 1. By using formula (50) and using results given in Cases 4 and 5 check whether relationship between discount factors for different currencies given by formula (51) stands for the following set of market data. Calculate the exact difference between discount rate for currency *Y* and *X*.

Table 15. Sample data for Exercise 1

Maturity T_M	IRS rate for the currency *X* $K_X^j(t,T)$ (%)	IRS rate for the currency *Y* $K_Y^j(t,T)$ (%)	Cross currency basis swap margin $b_{X,Y}^j(t,T)$ (%)	Implied difference between discount rates $s_{X,Y}^D(t,T) \cong K_Y^j(t,T) - +K_X^j(t,T) - b_{X,Y}^j(t,T)$ (%)
9M	2.60	2.40	0.20	–0.40
12M	2.80	2.60	0.40	–0.60

[19] The discount rate was calculated by using the following formula: $L_X^D(t,T) = -\ln[P_X^D(t,\boldsymbol{T})]/T$. For example for 9M tenor for currency *X* 2.50% = –ln(0.9817) / (3 · 90/365).

6.4.3. Building the forward curve

The formula (14) for pricing the floating leg of the IRS transaction requires the forward curve used to project the future level of the market reference rate connected with the transaction:

$$PV\_floating(t) = N \cdot \sum_{j=1}^{M} \left[\left(T_j - T_{j-1}\right) \cdot F\left(t, T_{j-1}, T_j\right) \cdot P\left(t, T_j\right) \right].$$

In this section, we will describe the algorithms for forward curve construction under the existence of collateral agreements in the new multi-curve framework.

The existence of non-zero tenor basis swap (TBS) margins has been discussed in the literature since the mid-1990s, see: (Fruchard et al., 1995; Tuckman & Porfirio, 2003; Boenkost & Schmidt, 2005; Flavell, 2006; Henrard, 2007). However, in the pre-crisis period, the observed TBS and CCBS margins were close to zero and had no significant impact on the pricing of interest rate derivatives. The abovementioned works were mainly theoretical without much effect on the observed market practice. As a result of the segmentation of the interest rate market observed after the financial crisis, the articles referred to above became the basis for new methodologies used in the construction of interest rate curves.

Below we present how the market quotes of TBS can be incorporated into the bootstrapping algorithm to build multiple forward curves for different tenors (e.g. 1M, 3M, 6M, 12M), taking into account their relationship implied by market quotes of TBS transactions.

Under the assumption that the forward rates for maturities up to T_N are known, the formula for the forward rate $F_X^j\left(t, T_{N-1}, T_N\right)$ for the period from T_{N-1} to T_N for currency X looks as follows:

$$F_X^j\left(t, T_{N-1}, T_N\right) = \frac{1 - P_X^D\left(t, T_N\right)}{\left(T_N - T_{N-1}\right) \cdot P_X^D\left(t, T_N\right)} - \\ + \frac{m_X^{j,D}\left(t, T_N\right) \cdot \sum_{i=1}^{N} \left[\left(T_i - T_{i-1}\right) \cdot P_X^D\left(t, T_i\right) \right]}{\left(T_N - T_{N-1}\right) \cdot P_X^D\left(t, T_N\right)} - \\ + \frac{\sum_{i=1}^{N-1} \left[F_X^j\left(t, T_{i-1}, T_i\right) \cdot \left(T_i - T_{i-1}\right) \cdot P_X^D\left(t, T_i\right) \right]}{\left(T_N - T_{N-1}\right) \cdot P_X^D\left(t, T_N\right)}, \qquad (52)$$

where:

$m_X^{j;D}(t,T_N)$ – margin of the TBS transaction between the reference rate j for which the forward curve is constructed and the discount rate for currency X.

The abovementioned formula can be simplified when quotes for IRS transaction for the reference rate j are available:

$$F_X^j(t,T_{N-1},T_N)=\frac{K_X^j(t,T_N)\cdot\sum_{i=1}^{N}\left[(T_i-T_{i-1})\cdot P_X^D(t,T_i)\right]}{(T_N-T_{N-1})\cdot P_X^D(t,T_N)}-\frac{\sum_{i=1}^{N-1}\left[F_X^j(t,T_{i-1},T_i)\cdot(T_i-T_{i-1})\cdot P_X^D(t,T_i)\right]}{(T_N-T_{N-1})\cdot P_X^D(t,T_N)}, \quad (53)$$

where:

$K_X^j(t,T_N)$ – fixed rate for the IRS transaction in currency X in which the fixed rate is exchanged for the reference rate j for which the forward curve is constructed.

The formula (52) can be derived by using pricing formulas (12) to (14) and reorganizing the equations to get the last forward rate (i.e. one covering the period from T_{N-1} to T_N).

Case 8

Building the forward curve using formula (53). As discount factors for currency X we will use discount factors obtained in Case 4.

Table 16. Building the forward curve [A]

Maturity T_N	**IRS rate for the currency X $K_X^j(t,T_N)$ (%)**	**Discount factor for the currency X $P_X^D(t,T_N)$**	**Days**	$\sum_{i=1}^{N}\left[(T_i-T_{i-1})\times P_X^D(t,T_i)\right]$	**A** $\frac{K_X^j(t,T_N)\cdot\sum_{i=1}^{N}\left[(T_i-T_{i-1})\cdot P_X^D(t,T_i)\right]}{(T_N-T_{N-1})\cdot P_X^D(t,T_N)}$
3M	2.20	0.9951	90	0.2454	2.20% = 2.20% · 0.2454 / (0.9951 · 90/365)
6M	2.40	0.9890	90	0.4892	4.81% = 2.40% · 0.4892 / (0.9890 · 90/365)
9M	2.60	0.9817	90	0.7313	7.85% = 2.60% · 0.7313 / (0.9817 · 90/365)
12M	2.80	0.9733	90	0.9713	11.33% = 2.80% · 0.9713 / (0.9733 · 90/365)

Table 17. Building the forward curve [B]

Maturity T_N	**Discount factor for the currency X** $P_X^D(t,T_N)$	**B** $\frac{\sum_{i=1}^{N-1}\left[F_X^j(t,T_{i-1},T_i)\cdot(T_i-T_{i-1})\cdot P_X^D(t,T_i)\right]}{(T_N-T_{N-1})\cdot P_X^D(t,T_N)}$
3M	0.9951	0.00%
6M	0.9890	2.21% = 2.20% · 90/365 · 0.9951 / (90/365 · 0.9890)
9M	0.9817	4.85% = (2.20% · 90/365 · 0.9951 + 2.60% · 90/365 · 0.9890) / (90/365 · 0.9817)
12M	0.9733	7.92% = (2.20% · 90/365 · 0.9951 + 2.60% · 90/365 · 0.9890 + 3.00% · 90/365 · 0.9817) / (90/365 · 0.9733)

Table 18. Building the forward curve

Maturity	**A (%)**	**B (%)**	**Forward rate** $F_X^j(t,T_{N-1},T_N)$ **[A-B] (%)**	**Discount factor implied from forward rate**[20]
3M	2.20	0.00	2.20	0.9946
6M	4.81	2.21	2.60	0.9883
9M	7.85	4.85	3.00	0.9810
12M	11.33	7.92	3.41	0.9728

In case of non-zero TBS margin, the difference $s_X^{j,D}(t,T)$ between the discount rate and the *j-th* reference rate can be approximated by the following formula:

$$s_X^{j,D}(t,T)=L_X^j(t,T)-L_X^D(t,T)\cong m_X^{j,D}(t,T). \tag{54}$$

The difference can be directly approximated by the TBS margin.

Case 9

Calculating the difference between discount rate and *j-th* reference rate. As discount factors for currency *X* we will use discount factors obtained in Case 4. As discount factors and forward rate for reference rate *j* for currency *X* we will use discount factors obtained in Case 7.

[20] The discount factor was calculated by using the following formula: $P_X^j(t,T_i)=\frac{P_X^j(t,T_{i-1})}{1+F_X^j(t,T_{i-1},T_i)\cdot(T_i-T_{i-1})}$. For example for 6M tenor: 0.9883 = 0.9946 / (1 + 2.60% · 90/365).

Table 19. Difference between rates for *j-th* reference rate and discount curve[3]

Maturity T_M	Discount factor for the currency *X* $P_X^D(t,T)$	Discount factor implied from j-th forward rate $P_X^j(t,T)$	Discount rate for the currency *X*[21] $L_X^D(t,T)$ (%)	Discount rate for j-th forward rate $L_Y^j(t,T)$ (%)	Difference between rate for discount curve and *j-th* reference rate (%)
3M	0.9951	0.9946	2.00	2.20	0.20
6M	0.9890	0.9883	2.24	2.39	0.15
9M	0.9817	0.9810	2.49	2.59	0.10
12M	0.9733	0.9728	2.74	2.79	0.05

Table 20. Implied difference between rates for *j-th* reference rate and discount curve

Maturity T_M	OIS rate for the currency *X* $K_X^D(t,T)$ (%)	IRS rate for the currency *X* $K_X^j(t,T)$ (%)	TBS margin $b_{X,Y}^j(t,T)$ (%)	Implied difference between discount rates $s_X^{j,D}(t,T) \cong m_X^{j,D}(t,T)$ (%)
3M	2.00	2.20	0.20	0.20
6M	2.25	2.40	0.15	0.15
9M	2.50	2.60	0.10	0.10
12M	2.75	2.80	0.05	0.05

Exercise 2. By using formula (53) and using results given in Cases 4 and 9 check whether relationship between discount factors for discount rate and *j-th* reference rate given by formula (54) stands for the following set of market data. Calculate the exact difference between *j-th* reference rate and discount curve.

Table 21. Sample data for Exercise 2

Maturity T_M	OIS rate for the currency *X* $K_X^D(t,T)$ (%)	IRS rate for the currency *X* $K_X^j(t,T)$ (%)	TBS margin $b_{X,Y}^j(t,T)$ (%)	Implied difference between discount rates $s_X^{j,D}(t,T) \cong m_X^{j,D}(t,T)$ (%)
3M	2.00	2.40	0.40	0.40
6M	2.25	2.55	0.30	0.30
9M	2.50	2.70	0.20	0.20
12M	2.75	2.85	0.10	0.10

[21] The discount rate was calculated by using the following formula: $L_X^D(t,T) = -\ln\left[P_X^D(t,\boldsymbol{T})\right]/T$. For example for discount curve 9M tenor 2.49% = –ln(0.9817) / (3 · 90/365).

6.5. Summary

The turmoil in the financial markets has shown that credit and liquidity issues are crucial in pricing financial products. The fundamental assumptions that have driven our practices in the interest rate market for many years needed to be challenged. In this chapter, we have provided a short outline how the risk-free yield curve differs from the discounting curve used before the 2007-2009 crisis. Consequently, methodologies and algorithms used in pricing should be reconsidered to take into account recent market developments.

In view of the above, we demonstrated a current market practice for multiple interest rate curve bootstrapping, each homogeneous in the underlying rate tenor and collateral arrangement, which takes into account market information carried by basis swaps between different tenors and currencies.

Below we present some key literature for reader eager to further study the topics covered in this chapter.

Further readings

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Numerical methods

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CHAPTER 7
HEDGE FUND STRATEGIES

Introduction

Hedge funds are known for being one of the most efficient but risky financial innovations in the world. They belong to the group of alternative investments. Contrary to traditional investment funds, they are offered only to selected investors who are characterized by high level of assets and sufficient degree of experience allowing them to allocate capital in investments not controlled by governmental institutions of financial market supervision.

The history of hedge funds dates to 1949 when Alfred Winslow Jones, a sociologist and former editor of *Forbes* magazine, launched the equity fund called A.W. Jones & Co. The fund had a structure of a limited partnership company (in which Jones was a general partner with the initial capital of 40% coming from his personal wealth), so it was able to avoid restrictive regulations of U.S. Securities and Exchange Commission (SEC) and be flexible in portfolio construction. The investment strategy assumed a combination of long positions in undervalued stocks and short positions in overvalued ones. In other worlds, the fund was based on a simple rule: "buy cheap and sell expensive." The positions were leveraged, so the proceeds from short selling were financing the purchase of additional long positions. The strategy was an innovation at that time, so in order to attract investors A. W. Jones gave up charging the management fee. Instead, he was charging 20% of realized profit. The concept–developed after some years into diversified multi-manager fund–was very profitable and giving A.W. Jones and his investors long-term abnormal returns. His success described by Carol J. Loomis in *Forbes* magazine in the article titled *The Jones nobody can keep up with* (Loomis, 1966) attracted the attention of the world of finance to hedge funds. Since then–despite of its ups and downs–the industry has grown intensively. Today it is a very important part of the global business at the forefront of financial innovation. A great portion of this innovation is connected with the use of derivatives in hedge fund strategies which are perfectly known by financial engineers.

The further text of the chapter is divided into three sections. The first two aim at explaining the essence of hedge fund business. Section 2. introduces the definition and describes the attributes that are unique to hedge funds. Section 3. classifies and characterizes the strategies of hedge funds, especially those that use derivatives on a daily basis. Section 4 presents a few examples of how derivatives are used in those strategies. They are hypothetical but they bring this issue closer and introduce the reader to how hedge fund managers earn absolute returns.

7.1. Definition and attributes of hedge funds

Although the term "hedge fund" has become widespread, it is a misname used by Carol J. Loomis in her article about Alfred Winslow Jones. There are two facts that decide about it. First, while describing his fund, A.W. Jones never used the word "hedge" (*nota bene* such a noun was not recognized by English dictionary at that time, it was created by C. J. Loomis) but the adjective "hedged" to distinguish it from other funds, in which the stock market positions were not *hedged*. Secondly (which is even more important), in reality hedge funds, which aim at earning absolute returns, take various risks (not only a market risk) that cannot be *hedged* (Ineichen & Silberstein, 2008, p. 10-11). This should exclude the use of the name "hedge(d) fund" by these institutions. However, the success of A. W. Jones' fund and its accompanying publicity made this misname a "real" name which started to be used in practice and science permanently.

Due to the lack of precise legal definition and huge diversity of hedge fund strategies **there is no uniform definition** of the term "hedge fund" in the lit-

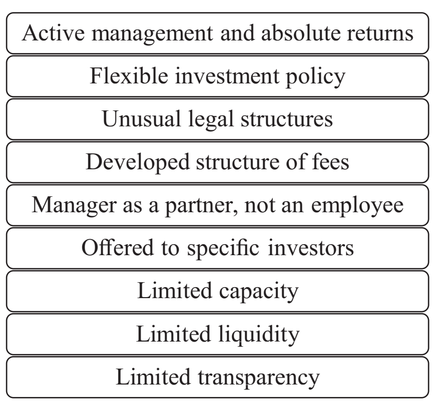

Figure 1. Attributes of hedge funds

erature: examples of different definitions are presented e.g. by Lhabitant (2006, p. 25) or Perez (2011, pp. 21-25). Therefore, the most common way of distinguishing them from other types of investment funds is to name their *attributes*. They are presented in Figure 1.

7.1.1. Active management and absolute returns

Mutual funds, which are traditional (or classical) investment funds, are managed passively or actively and try to earn returns higher than some benchmark (i.e. *relative returns* which might be positive or negative, depending on the situation on financial markets). Hedge funds work differently. They are always **managed actively because they have an investment goal of *absolute returns*** – returns that are supposed to be positive and independent of the situation on financial market (and hence a market risk). Hedge funds focus not on the risk of beta, but on so-called *alpha*. From the statistical point of view alpha is an absolute term from the linear regression modeling the efficiency of investment fund management, which is proposed by M. Jensen (1968, 1969) and based on the Capital Asset Pricing Model (CAMP):

$$\left(R_t - R_f\right) = \alpha_t + \beta_t\left(R_M - R_f\right) + \varepsilon_t,$$

where:

α_t is alpha coefficient,
R_t – a fund return in time *t*,
R_f – risk-free rate in time *t*,
β_t – beta coefficient of a fund in time *t*,
R_M – a return on market portfolio (benchmark) in time *t*,
ε_t – an error term.

Hedge fund managers focus on reaching a positive value of alpha coefficient ($\alpha > 0$) which represents the skills of the managers to perform superiorly and above average (the average performance is alpha neutral ($\alpha = 0$) and is represented by a passive portfolio; $\alpha < 0$ means that the manager's performance was inferior, so he underperformed)[1]. Only when $\alpha > 0$ their performance has *added value* and is concerned as *absolute* or *abnormal* and managers themselves are called *alphas*.

To achieve their investment goal hedge fund managers use three types of approaches to active management: directional, non-directional and a hybrid one.

[1] An interesting discussion on *alpha* as a source of risk management and the difference between *alpha* and *beta* see Lo (2010).

Directional approach is based on dynamic betting on the direction in which the markets will move and on taking long *or* short positions to capture the increases or decreases of the security prices that occur. The directional approach concentrates on searching for market risk and offering significant reward for it.[2] It is represented by strategies such as *global macro*, *sector* or *long-short equity*.

Non-directional approach is the opposite of the market timing approach. Managers use here structural market anomalies and build on this basis a diversified portfolio of arbitrage opportunities (with help of derivatives) that bring the added value. In this case, they take long *and* short positions in comparable financial instruments to add value and eliminate market risk. This approach is used, for example, in the case of *fixed income arbitrage* or *market neutral* strategy.

In addition to these, some managers (for example in event-driven strategies) also use a *hybrid approach*. In this case, they try to protect themselves from the market risk as much as possible; however, they are never able to completely eliminate it. As a consequence, this approach is characterized by greater volatility than the non-directional approach, but smaller than the directional approach (Fung & Hsieh, 1999, pp. 321-322).

7.1.2. Flexible investment policy

Hedge funds seek absolute return through the flexible investment policy, which depends on the type of investment strategy developed and implemented by a hedge fund manager (or managers). Generally, the managers have the freedom to choose investment styles, asset classes or investment techniques that they will use to achieve the assumed investment objective. In particular, they may (separately or simultaneously) combine long and short positions, concentrate rather than diversify their investment portfolio, borrow or leverage their portfolios, invest in illiquid or unlisted financial instruments, trade derivatives, hedge against declines of security prices on a given market or use short selling. They can also change strategies or markets if they see a chance for higher return in it (Lhabitant, 2006, p. 26).

The fact that hedge fund managers have such a rich spectrum of investment tools and structures at their disposal does not mean that they abuse it (this concerns especially leverage; which by many is believed to be too high and in reality is very reasonable; see Ang, Gorovyy, & van Inwegen, 2011; or Petroff, 2015). However, a flexible investment policy subjects the fund to a greater *manager risk* which depends on the skills and risk tolerance of the manager and his

[2] Note that market timing approach in hedge funds is different from market timing approach used by mutual funds, as it concerns not only long, but also short positions on various (not only traditional) segments of the financial market.

team (Lhabitant, 2006, p. 26). From the point of view of a potential client of hedge funds, it is important to firstly use the services of the manager, who will have the skills to properly use the available financial tools and techniques, and secondly, if possible, to diversify the portfolio of managers (Jaeger, 2003, p. 5), which contributes to reducing the risk of managing a hedge fund.

7.1.3. Unusual legal structure

Hedge funds are financial institutions that have an unusual legal structure. It is intended to be the subject to as few legal and tax regulations as possible and be suited to the nature of their business. In particular, this involves the possibility of:

- making various types of transactions (including opening long and short positions on any financial instruments, use of all possible derivatives, leverage and short selling) on various local financial markets around the world without any legal or organizational restrictions that could potentially "jeopardize" those transactions;
- optimizing the taxation of positive returns generated by a hedge fund – the scale of taxation of capital gains directly affects the net profitability of both investors and a hedge fund itself;
- avoiding the necessity of periodic reporting to any financial market regulator regarding the details of a hedge fund strategy.

When determining the legal form of a hedge fund, attention is also drawn to who will be its investors and where the fund will be registered. Accordingly, hedge funds use three types of legal structures that can be further subdivided according to their place of registration. They are presented in Figure 2.

Hedge funds are generally *limited partnerships* or *limited liability companies*. Both legal structures are recognized in the legal systems all over the world (though–depending on the jurisdiction of a given country–they may vary in de-

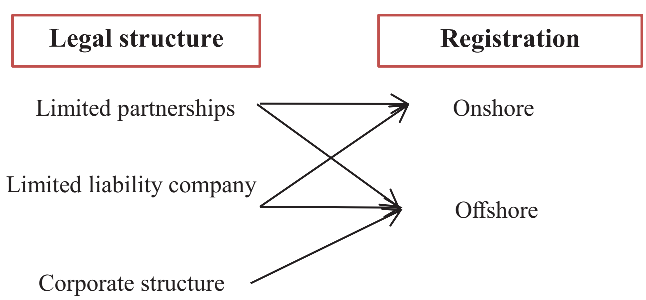

Figure 2. Legal structure and registration of hedge funds

tail). The main similarity between them is that both are separate legal entities in which taxation of earned returns does not burden the hedge fund itself (which, however, is obliged to report them), but is transferred directly to investors (so called *flow-through taxation*). The main difference between them is that:

- there are two types of partners in a limited partnership: a general partner and limited partners. A general partner is a fund manager responsible with his own wealth for debts and liabilities of a fund; limited partners are other investors who are responsible for the debts and liabilities of the fund only up to the amount of capital they invest in it;
- a limited liability company is owned by members (they can be both natural persons, i.e. individual investors and fund managers, as well as legal persons: institutional investors). Their responsibility for the debts and liabilities of a fund is limited to the capital invested.

Hedge funds are also divided according to the place of their registration. As in the case of legal structures, the division into *onshore* and *offshore* funds comes from the USA. Originally, onshore funds had their headquarters in the United States, and offshore funds outside of this country, mainly on islands – *tax heavens* with a close territory to the USA. Today the list of onshore countries contains also Great Britain and Japan. The list of offshore countries includes the Canary Islands, British Virgin Islands, Bermuda etc. as well as the so-called *offshore financial centers* (OFC) in Luxembourg, Dublin, Dubai, Singapore or Hong Kong. Very often an offshore country is only a place of registration of a fund. In reality its manager and all infrastructure necessary to make transactions (e.g. brokerage houses in which they buy/sell securities or from which they lend securities for short sale, or investment banks that give them loans) have their actual residence elsewhere–usually in one of the world's financial centers, i.e. New York, Chicago, London or Tokyo.

Due to tax benefits many offshore funds have a *corporate structure*, i.e. a joint stock company or its equivalent, or possibly a limited liability company (offshore limited partnership companies are very rare), often–unlike onshore funds–being open-end (and not closed-end) funds.

7.1.4. Developed structure of fees

One of the most important attributes of hedge funds are the **fees** they charge. Their structure is presented in Figure 3.

There are two types of fees which the activity of hedge funds is based on: **a management fee and a performance fee**. The first one is a fixed fee expressed as a percentage of assets under management per year. It covers salaries of a manager and her team (including analysts, sellers, traders, etc.) and other operational costs of a fund (including execution costs and other brokerage fees). A management fee is charged monthly, quarterly or annually, depending on the dates of subscription and redemption. Its amount depends on the complexity of the investment strategy implemented by a fund and its size as well as on the amount of fees charged by the competition (i.e. by hedge funds using the same or similar strategy). Most often it is 1-2% of net asset value (NAV) per annum, however, there are funds that charge their clients 3-4% of NAV p.a.

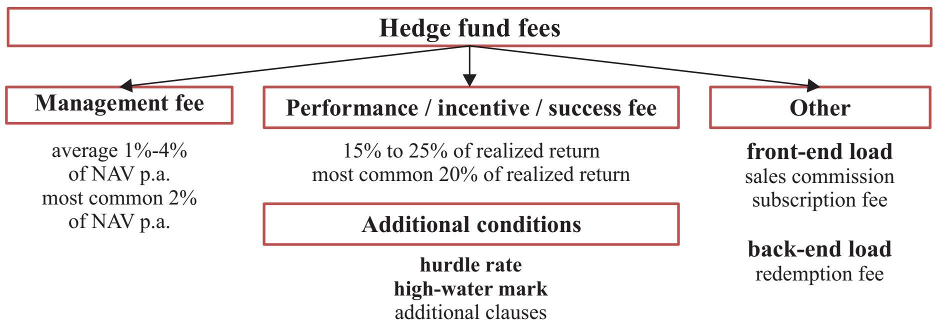

Figure 3. Structure of hedge fund fees

A *performance fee* (*incentive fee*, *success fee*) motivates and rewards a manager for earning an absolute return. It is paid after a given period (mostly a year) and it ranges generally from 15% to 25% (most often 20%) of the realized profit–its amount reflects of how much a manager values his skills and talent to manage financial assets (Lhabitant, 2006, p. 30).

Very often, this fee is charged only after meeting certain conditions regarding the return earned. The most common ones are a hurdle rate and a high-water mark. The *hurdle rate* indicates the minimum positive performance that a manager must earn in order to be allowed to charge a performance fee. It is a sort of a minimum acceptable return represented e.g. by the rate of open market operations or LIBOR or a fixed number (e.g. 3% and sometimes even 0%). The *high-water mark* states that any previous losses must be recouped by new profits before the performance fee is to be paid. Generally, it varies for each investor and is based on the maximum value on the investor's interest in the fund since his initial investment in it. This protects investors from paying a performance fee while they are just recovering from previous losses (Lhabitant, 2006, p. 30).

Apart from the hurdle rate and high-water mark, the payment of the performance fee may depend on additional clauses. To some extent they protect the interests of investors or a hedge fund manager in the event of losses. Information about them is always included in the offering memorandum of a fund.

The last group of fees paid by the investors are entrance and exit fees. *Front-end loads* include:

- a *sales commission*, which is a commission for a broker (it may range from 0% up to 5%, depending on a distribution channel and the value of assets that an investor invests in a hedge fund;
- a *subscription fee*, which concerns new participants and compensates high costs of launching a fund and its initial administration.

When exiting a hedge fund its participants are obligated to pay a *redemption fee*. By redeeming their shares investors somehow force a manager to close the most liquid positions and leave a higher number of illiquid positions in a hedge fund portfolio. The redemption fee allows to increase the liquidity of a hedge fund portfolio (the fee is automatically transferred to the fund's basket of assets). Usually, although its amount does not exceed 2% of the value of the redeemed capital, the more illiquid securities in a hedge fund portfolio and the shorter the terms of redemptions, the higher the redemption fee. In some cases, its movable scale is used depending on how long the investor maintains the given deposit and how early he notifies the manager about the redemption. In addition, when redemptions are high, managers may provide investors with additional shares in the fund instead of cash (Cottier, 1997, p. 28).

7.1.5. Manager as a partner, not an employee

One of the most important characteristics of hedge funds is that hedge fund managers are their partners, not their employees (as it is in traditional funds). They are founders of hedge funds who bring the initial capital on board. This capital is a part of their net worth, which in vast majority of cases is earned earlier in financial industry. Those managers represent both sides of financial markets: the buy side (ex-managers of conventional funds) and the sell side (former research analysts, brokers, traders and investment bankers). Apart from professionals directly related to financial services, we may find managers associated with this industry only indirectly or not at all. Alfred W. Jones, the father of hedge funds, a former sociologist and financial journalist, is such an example. The areas in which such hedge fund managers specialized earlier are e.g. mathematics, physics, nuclear engineering, biology, geology, computer science, chess or medicine. It is not uncommon that the best ideas for hedge fund strategies come from them. The experience from the fields completely different than financial industry allows them to look at it with a fresh eye and apply methods from other disciplines to the models which successfully predict absolute returns.

There is an unwritten rule that a hedge fund manager should engage a large part of his net assets in a hedge fund and preferably all of the assets he has reserved for investments in equity instruments, such as stocks. This means that the manager may retain, for example, government and municipal bonds or real estate bonds, but he should invest 100% of the capital he wants to invest in the stock market in a hedge fund he has set up. If he does not, investors should know about it. Firstly, because if he does not want to risk his wealth in a fund, why should they? Secondly, this may lead to unethical and illegal practice of *front running*.

Engaging manager's personal wealth in a hedge fund is supposed to be a motivation to rational approach towards investment risks and earning absolute returns. However, as Lhabitant (2006, p. 33) underlines, it can produce undesirable side-effects. On one side, at the beginning of his career, a hedge fund manager has little to lose. He may take too much risk, knowing that in case of a default, he can go back to traditional fund industry and recover quickly. On the other side, at the end of his career, when he is successful, he might resist from taking risks even though it is well remunerated.

7.1.6. Offered to specific investors

Another very important feature that distinguishes hedge funds from other types of investment funds is that they are offered to specific investors. Among the individuals we may find:

- HNWIs (high net worth individuals with the net wealth in cash of 1-3/5 million USD),
- very/extremely HNWIs with around 3/5-30/50 million USD,
- ultra HNWIs with more than 30-50 million USD.

In recent years, apart from HNWI, funds of hedge funds (FoHFs) or hedge funds that are new and determine the minimum value of investments at a low level, are also open to:

- wealthy individuals, with assets of 500,000 to 1 million USD,
- affluent individuals, with assets ranging from 100,000 to 500,000 USD.

However, it must be noted that often as those funds grow and increase the value of assets they manage, they raise the minimum threshold to enter the fund. Then the existing "small" individual investors have to redeem their shares in favor of new investors with more cash.

The second group of investors is represented by financial intermediaries like pension or investment funds, asset and wealth management companies or funds of hedge funds as well as non-financial institutions like endowments and charity or educational foundations. Both types of investors can be taxed or exempt from taxation on capital gains.

Wealthy individuals were the main clients of hedge funds up to the 1990s of the XX century. However, the favorable situation on the capital markets in the

1980s and 1990s, which caused significant cash surpluses among financial intermediaries, as well as globalization of financial markets (which allowed to build well-diversified international portfolios) and the possibilities of constructing sophisticated investment strategies based on financial engineering and new computer technologies, made hedge funds an important alternative also for institutional investors. Today they dominate the structure of clients of hedge funds.

7.1.7. Limited capacity

Hedge fund managers believe that "small is better"; therefore, they keep the fund capacity limited. This limitation is considered from three points of view:

- fund size,
- fund scope,
- number of key managers.

First of all, hedge funds limit the size of funds. In practice, regardless of the type of investment funds, the value of its assets is inversely correlated with investment results–the bigger the fund, the more difficult it is to invest money in affordable financial instruments, without affecting their price and liquidity, and the harder to earn positive returns. However, in case of conventional funds a rate of return is not so important because their managers are paid their bonuses based on the increase in the net asset value (NAV) of a fund and not on fund's performance, in hedge funds the latter is crucial both for the remuneration of managers and their reputation. Therefore, hedge funds focus on maintaining a small fund size which helps to achieve better performance and earn more from performance fees.

Limiting a size of a fund depends on the type of strategy a fund implements (different strategies are characterized by different levels of capacity; see for example *long only* versus *global macro* funds). As a rule, managers determine a certain amount of assets, beyond which they "close" a fund. It means not accepting new investors as fund participants and accepting additional capital only from investors who are already fund clients.

The limitation of the scope of hedge funds is related to the concentration of a given fund on a specific investment strategy, which is most often focused on a specific segment of the financial market. This results in specialization of fund managers, which allows them to process information from a given segment that may affect the prices of instruments quickly and easily, and to make transactions that are supposed to give a fund the highest possible return.

Finally, the limitation of hedge fund capacity is reflected in the fact that they are managed by a small number of key managers. Usually, a fund is managed by one or two people who create it and have a vision of its development. Such solution gives flexibility and speed of decisions made as well as productivity (which is lower in traditional funds, where investment decisions are made by an investment committee, that always lasts longer). On the other hand, it must be remembered that it brings additional manager risk.

7.1.8. Limited liquidity

Hedge funds are also characterized with limited liquidity. The investment goal of absolute returns requires from hedge fund managers implementing a long-term investment strategy. Very often this strategy involves using arbitrage opportunities on illiquid markets and financial instruments. In order to have a freedom of entering such investments and earning the above-average return, the managers limit the subscription and redemption possibilities of an investor and insist upon a minimum investment period. In *terms of subscription* they specify when the investors can purchase shares of a hedge fund. In closed-end funds it is only possible during the first issue. In open-end funds it is possible on a regular basis–usually once a quarter or once a month.

During some period after the purchase of fund shares investors cannot redeem them. This period is called a *lock-up period*, it is mandatory and, depending on a fund, lasts from 1 to 3 years. This is a minimum period during which an investor must keep her cash in a fund before she can withdraw it in accordance with the terms of redemption. The *terms of redemption* indicate when and under which conditions investors can redeem their shares. Generally, hedge funds allow it once a quarter, although less frequent dates (e.g. once every half a year or once a year), which is characteristic especially for funds operating mainly on illiquid markets and financial instruments, are not rare. If an investor wants to redeem his shares, he must give a hedge fund manager an advance notice. Usually the notice periods are from 30 to 90 days before the actual redemption (Lhabitant, 2006, p. 29) (the more frequent the redemption, the shorter the advance notice period).

In addition to the above, some hedge funds apply the following redemption requirements:

- *a hold-over provision*: when an investor wants to redeem 100% of his capital, a fund may keep 10% of this capita until the annual audit is finished,
- *a penalty fee*: this is a type of a redemption fee which is supposed to discourage investors from too early withdrawals; its value decreases with time, reaching, for example, after one year, the value of zero.
- *gate provisions*: are used by hedge funds to reduce the number of redemptions. Gate provisions are the limits defining the maximum percentage of the total assets of a fund, which may be withdrawn from it on a scheduled re-

demption date. The most commonly used limits are 20% of the value of assets in case of redemptions once a year or 10% of the value of assets when redemptions happen more frequently. The gate provisions allow the managers to increase exposure to less liquid assets or transactions without having to worry about a sudden liquidity crisis that may occur if several investors want to redeem their shares in one redemption date.

7.1.9. Limited transparency

The last but not least attribute of hedge funds is their limited transparency. Firstly, it is a result of their legal form, and, in many cases, a place of registration in offshore countries, as well as the fact that they are not subject to legislation on investment funds, so they do not have to disclose information about their performance or the allocation of their assets like traditional funds. Moreover, the disclosure of such information could be read by supervisory authorities of a given financial market as public marketing, and its use by such funds, e.g. in the USA, is forbidden. Secondly, it is connected to the fact that the details on the investment strategies implemented by hedge funds, including positions opened as well as transactions made or tools used must stay discrete. The opportunities to earn money could be very quickly copied by other managers, and, as a result, hedge funds would not be able to reach their performance goal. Therefore, "discretion" on details regarding the investment strategy is very important due to the potential success of managers.

The limited transparency of hedge funds was one of the reasons why for a long time information on the hedge funds market was difficult to access and selective. The experience of the last dozen or so years (especially the spectacular falls of such funds as LTCM) has led to progress in this matter. On one hand, today's investors demand a bit more accurate information, thanks to which they can conduct an effective due diligence, allowing them to choose hedge funds best suited to their expectations. On the other hand, the managers themselves, who care more about their interest and reputation, are not so reluctant to disclose the data on their activities (although these are rather aggregated than detailed).

Based on the above characteristics of hedge funds, we may formulate a definition of hedge funds, which is a summary of our considerations in this section. I propose that we define hedge funds as follows:

Hedge funds are alternative investment funds, which are offered mainly on non-public financial markets, and therefore are subject to limited legal regulations, and whose managers, using their skills and various investment tools and techniques, build investment strategies aimed at earning absolute returns, for which they are rewarded depending on their value.

7.2. Types and characteristics of hedge fund strategies

Hedge funds are internally a very diverse group of investment funds. Figure 4 classifies them according to their investment strategies. They can also be classified according to other criteria that allow a more detailed comparison of their strategies – see Lavinio (1999, p. 7):

- geographical area of activity (funds operating on the local, regional or global financial market),
- the level of fund diversification (more or less funds diversified),
- applying or not hedging of the positions,
- the level of liquidity or trading (funds from the smallest to the largest liquidity and trading financial instruments in the portfolio),
- using financial leverage (funds that use zero or very small to very high financial leverage),
- market exposure (funds from low to high correlation with the market risk),
- asset classes used in investment policy (value or growth stocks, fixed income instruments, etc.).

Although these criteria are very important, the essence of hedge fund investment strategies stays the most important.

7.2.1. Directional strategies

The first group of hedge fund strategies are directional strategies. They use directional approach to fund management. In this approach the managers, looking for return, speculate on the direction of changes in prices of securities in selected segments of the financial market. They make transactions on various securities, whose prices according to their predictions move in the same direction as the market and, additionally, hedge them with futures contracts. Managers invest here for a short time, quickly changing their view on the market. Some of them rely in their decisions on computer systems based on models using technical analysis that generate buy and sell signals. Others concentrate on valuation-based approach and use fundamental analysis. The rest combine both approaches, which makes the strategy more complicated, but at the same time allows for its precise implementation.

The most important directional strategy is **long/short equity** (also known as **equity hedge**). It was first used by the hedge fund pioneer Alfred Winslow Jones. It reflects an old and simple principle of "buy cheap and sell expensive". It consists in searching for under- or overvalued securities, and then on:

- buying (i.e. taking long positions) of those securities that are undervalued,
- selling (i.e. taking short positions) of those securities that are overvalued.

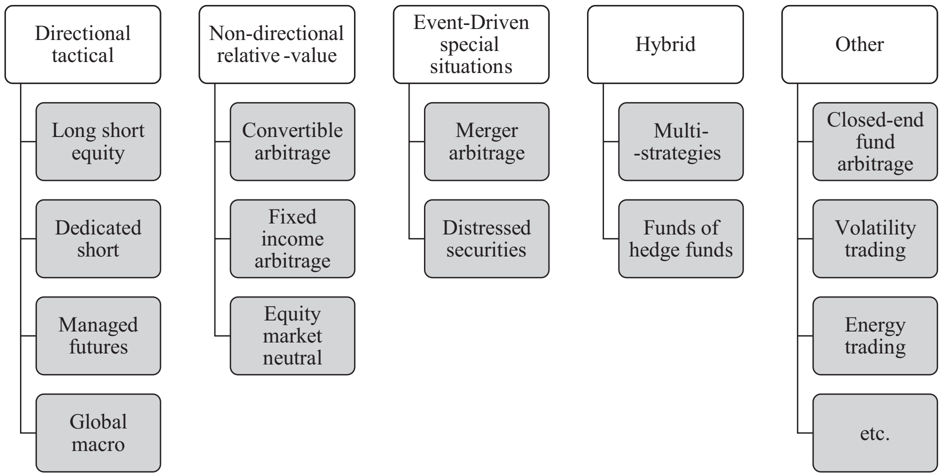

Figure 4. Strategies of hedge funds
Source: Based on (Stefanini, 2006, p. 14; Frush, 2007, pp. 19-22; Cottier, 1997, p. 117).

In this strategy a manager aims at reducing the overall risk of a fund portfolio by minimizing total market exposure and low correlation between the return on the fund portfolio and the return on the market portfolio. Portfolios of long and short positions are hedged mainly by the use of short selling. It is also not uncommon to use derivatives based on a given capital market index. In this situation the managers sell futures for market indices so that – reacting to market changes – they can quickly change the market exposure of their portfolio (Stefanini, 2006, p. 47).

When selecting securities for the portfolio of a long/short equity fund its manager may take the following positions:

- *straight long*, i.e. long positions in securities a fund manager considers undervalued,
- *straight short*, i.e. short positions in securities considered to be overvalued,
- *relative value*, i.e. related to *share class arbitrage*, including common stocks, preferred stocks and saving stocks, which do not give the right to vote, but are privileged in relation to common shares in terms of a dividend (such shares provide a minimum annual amount of a dividend that must be paid to their holders) or a liquidation of a company (they ensure a correspondingly higher amount of cash paid to their holders than to common shareholders); all three categories of shares differ in their valuation under different market conditions, which gives arbitrage opportunities,
- *pairs trades*, which is taking the opposite positions on a pair of shares of two substitute companies (e.g. long on Deutsche Telecom and short on Vodafone).

We must underline that hedge fund managers operate on the same financial markets that the managers of traditional funds. However, additionally the former use short selling, financial leverage and the manager's remuneration system for profit. Profit occurs when the prices of securities purchased by a manager go up and the prices of securities sold by him go down (in the opposite situation the strategy brings losses). Because there is an expectation of profit on both the buying and selling sides, quite often a long/short equity strategy is called a *double alpha strategy*.

Lhabitant (2006, p. 166-170) notes that the long/short equity strategy is beneficial because it allows to earn money in times of a bullish and bearish market and is characterized by a much lower (because more diversified) risk than a *long only* strategy applied in traditional investment funds. However, this strategy is not free from disadvantages, especially higher trading costs connected to higher turnover of securities in the portfolio than in passive strategies realized by traditional funds as well as net long bias, i.e. a higher long exposure than a short exposure. Even though, mainly thanks to its benefits and simplicity, this strategy is one of the most common among hedge fund managers.

The second type of directional strategies is *short selling* or **dedicated short**. This strategy is the opposite of *long-only* used by managers of traditional investment funds. Dedicated short is characterized by reverse (mirror) market exposure to traditional funds–the strategy brings positive return when the capital market is bearish and negative one when it is bullish. This means that dedicated short funds have a *net short exposure* (though during bullish time these funds use a sort of market timing and reduce the number of short positions in favor of short-term long positions). The strategy is characterized by high volatility of returns, but they are strongly negatively correlated with movements on the stock markets.

Generally, in selection of stocks to be sold, the managers focus on companies whose valuation, in their opinion, deviates from reality. The characteristics of such companies are presented by Stefanini (2006, p. 37) and Lhabitant (2006, p. 188). Those are for example:

- companies with deteriorated fundamentals because of *catalytic event* that has a negative impact on its activity in a short time (e.g. the announcement of lower than expected profits for a given period, accounting problems, problems with raising capital for investments). Before taking a short position a manager must be able to identify such an event. If it does not occur, such a company can be overvalued even for several years, then a manager will not be able to close the short selling transaction,
- companies with a high stock prices, but with weak financials (e.g. due to low cash flows, high P/E ratio or excessive leverage),
- companies with changes in the structure of equity,

- companies from industries in crisis caused by external factors,
- companies in which the management team fools investors (e.g. by using aggressive accounting practices, such as "accounting tricks" with options, pension funds, pro-forma data reports, and not actual ones),
- companies which destroy their value by maintaining low return on equity (ROE) and high P/E ratio or which place their liquid assets in investments, the return of which is lower than their ROE and thus erodes their profit structure,
- companies that are characterized by a high *insider selling*, i.e. a high number of shares that are sold by their top management.

When properly implemented, this strategy can turn out to be very profitable. One of the main arguments for its application is that there are still a lot of unused short selling opportunities on the capital markets. This is mainly due to the fact that the vast majority of asset managers (not only traditional, but partially also alternative ones) understand or have intuition and experience with entering long, not short positions. The same applies to individual investors who consider short selling to be too risky, and institutional investors who do not make it, because they cannot. As a consequence, brokerage houses and analysts focus on what to buy, not on what to sell, hence there is almost no competition in identifying the overvalued assets. However, it must be remembered that this strategy has a limited profit potential and unlimited loss potential. At the same time, the share of loss-making positions in the portfolio may increase relatively (Jaeger, 2003, p. 142). That is why the appropriate portfolio diversification that contributes to the reduction of the total market exposure is so important (Lhabitant, 2006, p. 188).

When considering a long/short equity and dedicated short strategies, it is important to mention *total net market exposure*. It is equal to the difference between the sum of the weights of the long positions and the absolute value of the sum of the weights of short positions (Stefanini, 2006, p. 49):

$$TME_{net} = \sum_{i=1}^{L} w_i - \left| \sum_{i=1}^{S} w_i \right|,$$

where:

TME_{net} – total net market exposure; the weight of a given share in a hedge fund portfolio expressed as a percentage of the fund's net asset value (NAV),

L – number of long positions in this portfolio,

S – number of short positions in this portfolio.

When:

$$\sum_{i=1}^{L} w_i > \left|\sum_{i=1}^{S} w_i\right|, \text{ we say that } TME_{net} \text{ is long } (\textit{net long}),$$

$$\sum_{i=1}^{L} w_i < \left|\sum_{i=1}^{S} w_i\right|, \text{ we say that } TME_{net} \text{ is short } (\textit{net short}).$$

In addition to the net value of the total market exposure of long/short equity and dedicated short funds, its gross value, given by the following formula, is often calculated:

$$TME_{gross} = \sum_{i=1}^{L} w_i + \left|\sum_{i=1}^{S} w_i\right|.$$

The *gross market exposure* value tells us how much cash the manager actually risks. The differences between the concepts of net and gross exposure are shown in example 1.

Example 1

Net and gross exposure in a long/short equity strategy

Let us assume that the long/short equity fund has only two shares in its portfolio: the long position is 70% of the NAV of the portfolio and the short one is 50%. The net exposure will be 20% (70% – 50%). The gross exposure will be 120% (70% + 50%). This means that the fund has open positions of 120% of its net asset value, which means that it has used leverage of 20%

Source: (Stefanini, 2006, p. 50).

Because the NAV is based on equal weights of individual elements of a portfolio so it rejects their sensitivity to market changes, it is not sufficient to determine the portfolio's exposure to systematic risk. That is why we use the concept of net exposure adjusted by the beta factor (called *beta adjusted market exposure*):

$$TME_{\beta} = \sum_{i=1}^{L+S} w_i \cdot \beta_i .$$

Net market exposure equal to zero does not imply an exposure corrected by a beta factor equal to zero. The first measure (net market exposure) is static, the second one (beta adjusted market exposure) dynamic, hence it has a variable value, sensitive to market changes. It is shown in example 2.

Example 2

Net exposure and net exposure adjusted by β

Continuing Example 1, let's assume that the beta factor of the long position is 0.5, and the beta factor of the short position is 1.5. Net position adjusted for beta is:

$$80\% \cdot 0.5 - 40\% \cdot 1.5 = -20\%.$$

Source: (Stefanini, 2006, p. 51).

Based on examples 1 and 2 we can see that initial positive net market exposure of 40% may turn out to be negative (–20%), when the sensitivity of different security prices to market changes are concerned. Due to the fact that long/short equity funds select securities from various industries with different liquidity, as Stefanini (2006, p. 51) suggests, it is always necessary to monitor the net exposure adjusted by the beta factor, as well as the liquidity of the portfolio, which can be measured e.g. as the average number of days needed to liquidate the portfolio without affecting the prices of securities held in it.

The next two strategies that are directional include managed futures and global macro. They are similar in being directional and investing mostly in futures contracts listed on capital markets worldwide. Both of them are also **tactical**, since their managers exploit market inefficiencies caused by: (1) market participants not motivated by economic profit (like central banks); (2) hedgers wanting to lay off risk; and (3) abnormal reactions of other market participants, particularly in times of market distress. The primary difference between them is that opposed to global macro strategies, where investment decisions are made by a human, managed futures strategies are realized by the use of computerized models that automatically make trading decisions and a manager only periodically re-adjusts the trading model assumptions about parameters.

Managed futures funds appeared in the United States in early 1970s. At that time, as a result of the regulatory separation between the occupation of a broker and an investment manager of futures contracts, a third occupation of a professional money manager emerged. Since the futures contracts existed only for commodities, such managers were known as Commodity Trading Advisors. With time this term evolved and now CTAs manage not only commodity futures contracts, but also (if not mainly) non-commodity futures contracts, e.g. on currencies, interest rates or stocks and bonds indices. Today the funds they manage are named interchangeably **managed futures**, **trading funds**

or **CTAs**. The managers apply in their models technical or fundamental analysis and look at the evolution of price patters in two ways:
- systematic (technical) traders use quantitative models to identify and implement trading opportunities,
- discretionary (fundamental) traders use human judgement and fundamental inputs.

The traders use two investing approaches shown in Figure 5. The characteristics of their trade implementations are:
- exposure to four major asset classes: currencies (e.g. euro, Mexican peso, British pound), interest rates (treasury notes and bonds, long gilt in the UK), commodities (energy: Brent Crude, heating oil, natural gas; metals copper, nickel, zinc and agriculture: coffee, cotton, wheat) and securities (e.g. stock indices: S&P 500, FTSE, DAX, Nikkei),
- the least constrained of all money managers,
- long and short exposure,
- time horizons varying from intra-day to several months.

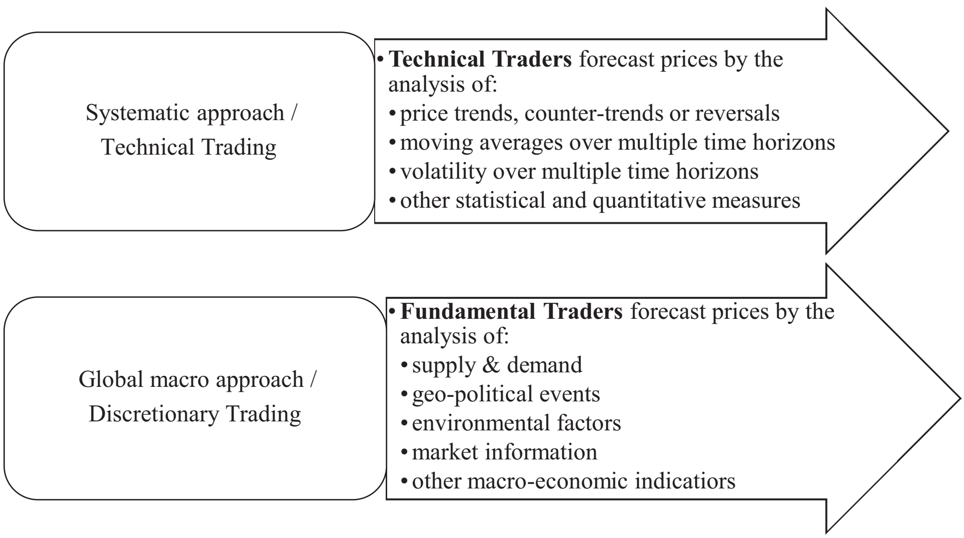

Figure 5. Investing approaches in managed futures funds

Unlike most hedge fund strategies, managed futures funds can be purchased not only on the private but also on the public market. This means that they are available to a relatively large group of investors (including small investors). In general, there are three ways to enter such funds:
- investors can entrust their capital with commodity pool operator (CPO) who creates an investment portfolio and employs one or more CTAs to manage it,

- investors hire one or more CTAs to invest their capital individually or employ a manager of managers (MOM) who looks for a suitable CTA for them,
- investors may purchase shares of a public commodity fund or a public futures fund, in the same way in which they buy stocks or bonds; such funds are intended for the widest group of capital market investors.

The strategy is generally offered in three investment structures: (1) managed accounts dedicated to institutional investors and HNWI; (2) managed futures funds; or (3) funds of managed futures funds which are a proposal for smaller individual investors.

The successful years of 1970s caused huge growth in assets under management of many managed futures and long/short equity funds. They were no longer able to trade securities without affecting their market prices so many of their managers made a *style drift* to the **global macro** strategy. It gave them a possibility to access different currency, commodity and treasury markets of high liquidity without the necessity of dealing with the *capacity* problem (Stefanini, 2006, p. 241).

Global macro funds are one of the most heterogeneous group of hedge funds but they have two things in common: they invest globally and they focus on macroeconomic trends and inefficiencies (e.g. significant changes in interest rates or exchange rates, important events in the country's economic policy or anomalies on the securities market that may be determined by economic changes) which they use to make abnormal returns. Most of global macro managers use a discretionary way of investing which is highly subjective, so their skills are of key importance to the success of this strategy. When constructing a portfolio, they use the approach that is called *top down*. In this approach a fund manager first selects the geographical area and the sector in which he will invest and then he analyzes the macroeconomic, political and market situation that affects the stock, currency, interest rate or commodity markets. On this basis he selects securities that he buys or sells in a short time (without analyzing the foundations of individual values). Managers using this method are generally referred to as *stock selectors*.

It is worth adding that *top down* is the opposite to *bottom-up*. The latter approach involves a deep fundamental analysis of individual company stocks and selecting those that managers want to buy or sell in a short time. Managers using this method are called bottom-up managers or *stock pickers* (Stefanini, 2006, p. 52). This approach is characteristic for long/short equity and managed futures managers and initially it was also used by managers who moved to global macro strategy. Therefore, by changing the strategy from long/short equity or managed futures into global macro, they also changed the method of selecting financial instruments to the fund portfolios.

Generally, global macro funds are directional. The managers look for opportunities to make a profit when the changes in macroeconomics, politics or the financial markets are to occur. It may be, for example, an adoption of a new law, change of the ruling party, signing an international agreement causing the flow of capital or lack of support for an economic project.[3] However, some of the funds may make convergence/divergence trades, in which managers bet on certain events that move the securities prices in different directions.[4] Global macro funds are also characterized by:

- large size (they must have the right amount of capital to be able to benefit from the possibility of earning an absolute return on a global scale),
- huge diversification; as Anson (2006, p. 18) notices, it can be both an advantage (it allows for the application of the presented strategy in the widest spectrum of instruments) and a disadvantage (it means lack of concentration on details, which mainly depends on institutional investors),
- a very high degree of the use of derivatives (options, futures and swaps), which involves high leverage (Lhabitant, 2004, p. 6); among the various types of hedge funds they are the most risky ones but also potentially the most profitable ones.

According to Hedge Fund Research in early 1990s global macro funds accounted for more than 70% of assets under management of the entire population of hedge funds. But around the new millennium (especially in times of the technology bubble of early 2000s) their share in the industry started to decrease dramatically and in the last 15 years it has been around 3%. The decrease in their number is certainly caused by a significant reduction of the possibility of achieving high profits (especially the last 5 years were difficult) (see: Kishan & Burton, 2017). Although the share of global macro funds in the hedge fund market has decreased, they are at the forefront of the largest hedge fund assets in terms of value.

[3] A typical example of this approach is the famous attack on the British pound made by George Soros, the manager of the Quantum fund, who predicted that the currency is overvalued and on September 22, 1992 made a short selling transaction worth $10 billion. This forced the Bank of England to withdraw the currency from ERM II, and brought Soros $1 billion profit. In his book Soros on Soros (Soros, 1995), the author stated that he knew that the German Bundesbank and the Bank of England would allow the pound to break out of the ERM II regime, and therefore decided to make a short selling of this currency. The decision on the execution of this transaction was made after a conversation with then Bundesbank head Dr. Helmut Schlesinger, whose answer to the question about such eventuality was unclear (Chorafas, 2003, p. 122); for the description of the transaction see Lhabitant, (2006, pp. 329-332) or Stefanini, (2006, pp. 243-244).

[4] An example of such transactions is short sale of the futures index for European shares and the purchase of shares on the US stock exchange. Another example is holding a short and long positions in the same local capital market (Chorafas, 2003, p. 123).

7.2.2. Non-directional or relative value strategies

The second group of hedge fund strategies is named **non-directional** or **relative value**. The managers of such funds identify inefficiencies in the valuation of related securities (e.g. stocks of companies from the same industry) and open the opposite positions, thereby neutralizing exposure to the market risk (i.e. trying to isolate alpha and achieve a beta ratio close to zero). In order to determine these inefficiencies, managers build complex quantitative models in which they use mathematical or statistical and fundamental or technical analysis, assuming that over time the market prices of mispriced securities will converge to its theoretical or intrinsic value (Lhabitant, 2004, p. 8). Additionally, since the inefficiencies in the prices are usually very small, these funds often use very large leverage to magnify returns.[5] It constitutes the most common and advanced use of derivatives.

The most known non-directional strategies of hedge funds are **convertible bond arbitrage**, fixed-income arbitrage and equity market neutral.

Convertible bonds are usually issued by young and dynamically growing companies with a low credit rating. They are fixed income securities which additionally give their holders an option to buy a fixed number of shares (mostly common stocks of the issuer or sometimes of another company from his capital group).[6] An important characteristic of convertible bonds is that usually the built-in option makes their market price undervalued in comparison to their theoretical value (equal to conversion value which is the market price of stocks into which a bond can be converted at a given time). That means that they have a low price in relation to the market price of the stocks which they can be converted into. This gives rise to arbitrage opportunities for hedge fund managers who take a long position on convertible bonds and a short position in these stocks or options for these stocks.

Hedge fund managers usually look for convertible bonds that have the following characteristics:

- high volatility of the underlying stocks, to profit from delta trading,
- high volatility and high convexity of convertible bonds to profit from gamma trading,
- good liquidity of bonds and easiness in borrowing underlying stocks,
- low conversion premium, because the bond is less sensitive to the interest rate risk and credit risk,
- they can be converted into stocks that do not pay out dividends or pay out low dividends, to avoid having to pay them out to stockholders from whom the shorted stocks were borrowed,

[5] However, this may have the unintended consequence of increasing the risk like in the famous case of Long Term Capital Management (LTCM) fund.

[6] So they are *plain vanilla*.

- issues with low implied volatility (Stefanini, 2006, p. 111).

In order to find such bonds and manage related risk types, convertible arbitrage funds build their strategies based on complex quantitative models that are used to assess the theoretical value of convertible bonds, calculate discrepancies between current prices of bonds and shares or calculating Greek values. The models also allow risk management and include the management of the market risk (that occurs when the equity hedge is not market-neutral) and the interest rate risk (which is related to possible changes in interest on convertible bonds during their life) as well as the credit risk (they use credit default swaps to hedge this risk), the hedge risk, the corporate event risk, the currency and liquidity risk or the *market corner* risk of sudden calls for the stocks to be returned within the short selling transactions. The risks can be minimized by due diligence, adequate hedging or diversification among issuers with different credit ratings, coming from different industries or operating in different countries (Stefanini, 2006, p. 133).

An important tool used in the strategy of convertible arbitrage is financial leverage. Its size depends on the size of long positions and investment objectives of a given hedge fund. It is usually 2 to 6 times the capital invested (Anson, 2006, p. 22). This is not much, especially in comparison with another arbitrage strategy–fixed income arbitrage, where the leverage reaches up to 1 to 20.

Fixed income arbitrage managers take offsetting positions in fixed income securities and their derivatives in order to exploit price inefficiencies between interest rate securities. Knowing these inefficiencies, they buy one fixed income security and simultaneously sell a similar fixed income security in order to hedge the market risk which the first security is exposed to. Usually, the two securities are related either mathematically or economically to such extent that when the market situation changes their prices move together in the same direction (either up or down). Generally, the difference between these prices is small but this is where the fixed income arbitrageurs can earn an absolute return. By buying and selling two related securities they hope to capture pricing discrepancies that will result in the price conversion of both of them over time (Anson, 2006, p. 23).

Fixed income arbitrage strategy is characterized by complex risks.[7] However, due to the fact that the market risk is neutralized and the interest rate risk, credit risk and other risks related to purchase and sale of fixed income securities are hedged by derivatives, the strategy is characterized by limited volatility. It depends on the duration of the investment and on the size of the financial leverage,

[7] Apart from market risks these are: interest risks, convexity risks, volatility risks, curvature risks, default/credit risks and prepayment risks in case of mortgage securities as well as liquidity risks, "flight to quality" risks, country risks, currency risks, counterparty risks, leverage risks or pricing risks.

which in fixed income arbitrage is large or very large–it multiplies small potential profits resulting from small differences in price spreads of individual security pairs. It can reach up to 20 times the capital involved in the strategy. Such leverage was almost a rule in the Long Term Capital Management which applied this strategy.

Transactions in this strategy concern treasury bonds (mainly of the United States, but also of other developed and developing countries), municipal bonds, corporate bonds, mortgage bonds, interest rate derivatives (including swap transactions) and all other fixed-interest instruments, both on a local and a global scale. Because the prices of fixed income instruments depend on various factors, hedge fund managers use sophisticated quantitative models run by powerful computers to identify mispricing on the market (Stefanini, 2006, p. 137). Just as it is important to build the model itself, it is also important to accurately identify the factors affecting these instruments. This requires very high qualifications from the managers, but only then they can earn absolute returns.

Fixed income arbitrage strategies vary widely and include many sub-strategies. The most frequently used ones are:

- *yield curve arbitrage*: a manager takes long and short positions at various points of the yield curve of government bonds, which approach the maturity date, so as to profit from inefficiently priced treasuries along these various points (which are different maturities). The mispricing leads to distortions of the yield curve, which brings opportunities for arbitrageurs. There are two types of a yield curve arbitrage:
  - *intra-curve:* if a manager trades securities of the same country, i.e. only within one given yield curve,[8]
  - *inter-curve:* if a manager makes transactions on instruments issued by governments from different countries, i.e. between two yield curves of two different currencies,
  - *on-the-run vs off-the-run Treasuries arbitrage:* arbitrage of treasury securities with different maturities (most often used for bonds, bills and notes issued by the US government). *On-the-run* securities are recently issued instruments which, therefore, have the highest liquidity on the US Treasury securities market. *Off-the-run* securities are instruments with similar characteristics but, due to the earlier issue date, they are no longer so liquid. As a result, there are discrepancies in prices of these securities (on-the-run treasuries have higher and off-the-run–lower prices, i.e. lower and higher yields respectively). The spread between the prices is not more than 1/2 or 1/4 pp, however, during uncertainty on the market, when investors transfer their capital to the most liquid treasury securities, it may

[8] See Stefanini (2006, p. 138) for further division of this strategy.

increase. Arbitrageurs earn on these spreads until maturity day, when the prices of these securities reach the theoretical value. It is worth mentioning that LTCM was specializing in this strategy;

- *mortgage-backed securities arbitrage*, which consists in taking the opposite positions on mortgage bonds issued in the United States and United States treasuries and profiting from differences in their valuation;
- *carry trades*, where a trader takes a long position in the security of higher yield and a short position in the security with lower yield (e.g. you can buy bonds with a higher yield than the cost of a loan taken for their purchase). Carry trade may be an *intra-curve arbitrage*, i.e. within one yield curve (when, e.g. for example, we borrow money at the price of a 3M interest rate to buy 10Y treasury bonds) or *inter-curve arbitrage*, i.e. between two yield curves (when, for example, we borrow dollars in the United States in order to buy Bulgarian bonds for Bulgarian lev–in this case it is necessary to estimate the currency risk). In the case of the carry trade the most important risk that can potentially arise is the risk that the price of the higher yield instrument will suddenly start to fall, which will cause a drop in the profit of this strategy.

Another strategy is **equity market neutral**. Managers concentrate here on constructing an investment portfolio that includes inefficiently valued and related securities (especially stocks, but also stock futures or options) which is market neutral. In other words its performance is not correlated to the market movements. Market neutrality can be understood as:

- *dollar neutrality:* neutrality of the values of long and short positions, e.g. in the US dollar. For example, if we have long positions worth 10 million USD in order to neutralize our portfolio, we should take short positions with the same value of 10 million USD. During our investment, when the value of one of the positions will change (e.g. the value of long positions will increase), in order to remain market neutral, we must increase the value of short positions – our portfolio is therefore neutral in terms of the value of the positions opened. Notice that in case of dollar neutrality our portfolio has a zero net market exposure,
- *beta neutrality*, also called *systematic neutrality*, means neutralizing systematic risk of a fund. It is reached when the values of beta ratios of long and short positions of a portfolio are equal:

$$\beta \text{ neutrality: } \beta \text{ of long positions} = \beta \text{ of short positions.}$$

To create a beta-neutral portfolio, a hedge fund manager first needs to estimate beta ratios of long and short positions, and then modify both parts of his portfolio so that beta ratios are the same. For example, if a beta coefficient of a long side of a portfolio is 1.6, and of a short side 0.8, then the beta of this portfolio will be 0.4 (50% · 1.6 – 50% · 0.8). So in order to neutralize this portfolio in terms of the beta coefficient, the size of short positions should be doubled, which means that every 1 USD of long positions corresponds to 2 USD of short positions. Only then the beta of this portfolio will be exactly zero and the systematic risk will be neutralized.

As previously, a portfolio should be monitored during the investment, because, for example, as the value of long positions increases or the value of short positions decreases, their beta coefficients change. For a portfolio to remain neutral in terms of the beta coefficient, the weight of individual securities in this portfolio should be adjusted over time:

- *Sector neutrality* means balancing the value of long and short positions of a portfolio in the same sector or industry. By doing so, hedge fund managers avoid risk of losses that may occur in some sector or industry. In addition, a portfolio stays neutral in terms of beta at the aggregate level.
- *Factor neutrality*: the most quantitative type of neutrality. It is used by managers who build multi-factor models in order to identify market risk associated with their portfolio, calculate the portfolio exposure to this risk and possibly neutralize it. The factors included in multifactor models can be divided into: microeconomic (related to a given company whose stocks may be included in a portfolio), mezzoeconomic (related to a sector in which a fund is involved) or macroeconomic ones (factors from a given local or global market). A portfolio is neutral in terms of factors when the betas concerning each of the factors from the model are zero. If they are not, a portfolio is not factor-neutral. Note that the possibilities of creating a factor-neutral portfolio are limited: the more risk factors a manager hedges away, the harder it is to neutralize his portfolio.

Similarly to long/short equity, the strategy of equity market neutral is called *double alpha*. However, we must underline, that making profits by neutralizing long and short positions is strictly dependent on the manager skills who must be very precise about the stock selection in his portfolio.

Usually, while managers of funds that neutralize the market risk benefit from short selling, they try to apply financial leverage to a very small extent. This is mainly due to the fact that the leverage cannot be hedged in any way on the market, which means that it cannot be neutralized. As a result, these funds are characterized by minimal credit risk (Anson, 2006, p. 32). Interestingly, despite much a lower risk than the market risk, the average returns on these funds are similar to the returns on market indices (McCrary, 2004, p. 23).

7.2.3. Event driven funds

Event-driven funds focus on exploiting security price inefficiencies that occur in the global marketplace due to corporate activities,[9] which are *catalytic events* because they bring a market price of a given company to a new value. Fund managers try to predict the outcome of a given event and its impact on the value of the company, as well as the moment when it is best to enter this investment. Those who do it correctly win and earn absolute returns. Event driven strategies are usually non-directional, leverage-free and have risk lower than the average on the capital markets. The most important strategies in this group are merger arbitrage and distressed securities.

Managers of the **merger arbitrage** strategy specialize in seeking opportunities for profit that arise from extraordinary corporate events such as mergers and acquisitions, leverage buy-outs or hostile acquisitions. Very often, this strategy is called *risk arbitrage* because its outcome depends entirely on the risk related to whether a merger or an acquisition will be finalized or not (Stefanini, 2006, p. 75).

In a **distressed securities** strategy managers concentrate on purchasing equity or debt securities listed on public markets or other instruments and trade receivables of companies threatened with bankruptcy, undergoing reorganization or having financial or operational difficulties. Hedge fund managers buying such securities are called *vulture investors*.

Such securities have very low liquidity and are ineffectively priced (actually their market price is close to default). Very often they are delisted from the public stock exchange and continue to be traded on the over-the-counter market (OTC). This is where their owners wishing to get rid of them short sell them to the vulture funds. Buying distressed securities at low prices allows hedge funds to increase the liquidity of their markets and make profits from the inability to appreciate the intrinsic value of these securities. They use two types of investment approaches in order to profit from such securities: a *passive* and an *active* approach. Passive vulture managers only purchase distressed securities and do not take over the control or participate in the restructuring process of the issuer's. He only waits for the security market price to increase to a fair value, and then sells it. Active vulture manager is very much involved in the business of a company that is his investment objective. Most often, he takes control of a company by buying a sufficient number of shares to take control of a company and its assets. Then he usually proposes a restructuring plan aimed at increasing

[9] Such as spin-offs, mergers and acquisitions, consolidation, liquidation, reorganization, bankruptcy, recapitalization, changes in the composition of the index or benchmark, sale or purchase of assets, discrepancies in the valuation of shares of various categories, contracts, legal disputes, investments in real estate or any other, most often unusual, events.

its value. Realization of profits in this approach takes longer than in the passive approach–usually a few years. The portfolios of this type of funds are also more concentrated than in the passive approach, where the portfolios of funds are more diversified (Lhabitant, 2006, pp. 230-231).

7.2.4. Hybrid and other funds

Hybrid funds concentrate on diversification of their portfolio among either different strategies (multi-strategy funds) or different managers (funds of hedge funds, FoHFs). Hybrid funds are a combination of different strategies from those mentioned above, however, most commonly within one group. This is supposed to be more efficient and profitable solution. High level of diversification makes them also available to a wider group of individual investors – not only HNWIs but also individuals with smaller net worth. We must be aware, however, that funds of hedge funds charge their clients double layer of fees – first on the level of funds acquired to a portfolio of a FoHF and then on the level of this FoHF.

All the strategies that do not belong to groups mentioned above are the "other" strategies. They use the most recent opportunities to arbitrage and therefore they are the most innovative. They are also believed to have a high potential of reaching an absolute return.

7.3. Examples of derivatives' use in hedge fund strategies

Hedge funds are able to earn abnormal returns mainly because they have many financial tools at their disposal. One of such tools are derivatives. They are used in all strategies that are hedged and/or leveraged and in all strategies which trade them to make profit. Below we can find some examples of the use of derivatives in hedge fund strategies. They are completely hypothetical and purely illustrative, but they show how hedge fund managers make money with the help of derivatives.

7.3.1. Covered call and put options sale in long / short equity

Generally a manager of a long/short equity fund takes long positions in undervalued stocks and short positions in overvalued stocks. He hopes that the prices of the stocks will move towards their equilibrium in a short time. However, sometimes there are no catalytic events on the market and the prices of stocks selected by a manager may stay mispriced for a long time. Then he can fall into the *value trap* (his undervalued stocks stay undervalued and/or overvalued stay overvalued for a long time). To improve the performance of his portfolio a manager can decide to do one of the two actions:

1) sell *out-of-the-money call options* to sell the owned stocks at the target price defined as the price at which a manager is willing to sell these stocks,
2) sell *out-of-the-money put options* to buy the sold stocks at the target price.

In this case a manager produces a "synthetic" catalytic event which is the option maturity. The examples of both situations with their outcome are presented in Table 1.

Table 1. Examples of using options in long / short equity strategy

		Sale of *out-of-the-money call options*	
		initial trade	**trade improving strategy performance**
		buying stocks of company A for 30 USD with the target price of 35 USD	selling European *out-of-the-money call options* with exercise price = target price of 35 USD + receiving option premium
Scenario at maturity of options		**outcome**	
1	market price of a stock A > 35 USD	profit of a manager = = market price of a stock A – 30 USD if he sells the stocks A	loss of a manager = = market price of a stock A – 35 USD because obligation to deliver stocks to a buyer of the call options at 35 USD
2	market price of a stock A = 35 USD	profit of a manager = = 35 USD – 30 USD = 5 USD per a stock A	profit/loss of a manager = 0 because obligation to deliver stocks to a buyer of the call options at 35 USD
3	market price of a stock A < 35 USD	profit for a manager if he sells stocks A as long as market price of a stock A > 30 USD loss of a manager when a market price > 30 USD	profit of a manager = = option premium because call options are not exercised

Table 1 – cont.

		Sale of *out-of-the-money put options*	
		initial trade	**trade improving strategy performance**
		short selling stocks of company B at 35 USD with the target price of 25 USD	selling European *out-of-the-money put options* with exercise price = target price of 25 USD + + receiving option premium
Scenario at maturity of options		**outcome**	
1	market price of a stock B < 25 USD	profit of a manager = = 35 USD – market price of a stock B if he buys back and closes his short position in stocks B	loss of a manager = = 25 USD – market price of a stock B because obligation to buy stocks from a buyer of put options at 25 USD
2	market price of a stock B = 25 USD	profit of a manager = = 35 USD – 25 USD = 10 USD per a stock B bought back	profit/loss of a manager = 0 because obligation to buy stocks B from a holder of the put options at 25 USD
3	market price of a stock B > 25 USD	profit of a manager if he buys back stocks B as long as market price of a stock B < 35 USD loss of a manager when a market price < 35 USD	profit of a manager = = option premium because put options are not exercised

Source: Based on (Stefanini, 2006, p. 64-65).

7.3.2. Volatility trading in convertible arbitrage

To remind the reader: convertible arbitrage funds take long positions in convertible bonds and then hedge the equity component of the bond by short selling the underlying stock or options on that stock. Equity risk in this strategy may be hedged by the short sale of the appropriate number of underlying stock determined by the *hedge ratio* which in this case is *delta*. Delta shows the degree of sensitivity of the value of a convertible bond to the price changes of underlying stock (Anson, 2006, p. 20). From an economic point of view, the delta allows to answer the question of how much the price of a convertible bond will change when its parity (the underlying stock price) changes by 1%. From the mathematical point of view, the delta is the first order derivative of the function of the convertible bond price to the underlying stock price.

Calculating *hedge ratio* means calculating the number of underlying stocks that must be sold short so the portfolio of convertible arbitrage fund is delta neutral (see example 3).

Example 3

Calculation of the hedge ratio using delta indicator

Let us assume that the convertible bond price is 10 000 USD, and the current market price of the underlying stock is 500 USD with a conversion premium of 50%, which means that the value of the conversion premium equals to 750 USD. Let's assume that the bond has a delta of 0.75. Then the hedge ratio is equal to:

$$(10000 \text{ USD} / 750 \text{ USD}) \cdot 0.75 = 10.00.$$

This means that in order to be able to hedge one purchased bond, a short sale of 10 shares should be made. Then the portfolio of the hedge fund will be delta neutral.

Source: (Stefanini, 2006, p. 113).

Convertible bonds, which trade at a low premium relative to their conversion price, are usually more correlated with the underlying stock price changes (they behave more like stocks than bonds). Consequently, in order to hedge equity risk included in the bond, a high delta is required. In the opposite situation, when bonds trade at high premium relative to their conversion price (they behave as fixed income securities), a lower value of delta is required. In this case, however, there is a greater interest rate risk than in the first case, when there is a higher equity exposure. The interest rate risk is managed by selling interest rate futures or swaps or other bonds. It should also be emphasized that hedging ratios of equity and interest risk are not static. Their values change in every moment, in which the prices of underlying stocks and interest rates change. Therefore, a hedge fund manager must continually adjust his deltas to ensure that his strategy remains intact (Anson, 2006, p. 58).

Example 4

Possible return in volatility trading in convertible arbitrage

A hedge fund manager buys 10 convertible bonds with a par value of 10 000 USD, a coupon of 4.5% and a market price of 9000 USD. The conversion ratio for the bonds is 18 and it is based on the current price of convertible bond and current price of underlying stock equal to 500 USD (9000 USD / 500 USD = 18). Delta of these bonds is 0.5. In order to hedge equity risk, a hedge fund manager must sell short the following number of underlying stocks:

$$10 \text{ bonds} \cdot 18 \text{ conversion ratio} \cdot 0.5 \text{ hedge ratio} = 90 \text{ shares of stock}$$

In such conditions, the arbitrage means buying 10 convertible bonds and selling 90 shares of stock. With the equity exposure hedged, a convertible bond is transformed into a traditional fixed income security with a coupon of 4.5%.

Additionally, the hedge fund manager earns interest on the cash proceeds received from the short sale of stock (known as *short rebate*) of 1.5%.

The total return on this strategy depends on the change of the prices of both the bond and the underlying stock.

Let's consider four scenarios, with previous conditions constant (no changes in hedge ratio during the holding period of one year):

- scenario 1, when the price of stock increases to 520 USD and the price of convertible bond to 9200 USD,
- scenario 2, where additionally to scenario 1, the fund is leveraged in 50% (so the manager purchased the convertible bonds with 45 000 USD of initial capital and 45 000 USD of borrowed money; let's assume that he borrowed the additional capital with his prime broker at a prime rate of 2,0%),
- scenario 3, when the price of stock decreases to 480 USD and the price of convertible bond to 8800 USD,
- scenario 4, where additionally to scenario 3, the fund is leveraged in 50% (so the manager purchased the convertible bonds with 45 000 USD of initial capital and 45 000 USD of borrowed money; let's assume that he borrowed the additional capital with his prime broker also at a prime rate of 2,0%).

Scenario 1

Appreciation of bond price:	10 · (9200 – 9000)	= 2000
Appreciation of stock price:	90 · (500 – 520)	= –1800
Interest on bonds:	10 · 10000 · 4.5%	= 4500
Short rebate:	90 · 500 · 1.5%	= 675
Total:		= 5375 USD

In % the total return on capital is: 5375 USD / 90000 USD = **5,9723%**.

Scenario 2

Depreciation of bond price:	10 · (9200 – 9000)	= 2000
Depreciation of stock price:	90 · (500 – 520)	= –1800
Interest on bonds:	10 · 10000 · 4.5%	= 4500
Short rebate:	90 · 500 · 1.5%	= 675
Interest on borrowing	2% · 45000	= –900
Total:		= 4475 USD

In % the total return on capital is 4475 USD / 45000 USD = **9,9445%**.

Scenario 3

Appreciation of bond price:	10 · (8800 – 9000)	= –2000
Appreciation of stock price:	90 · (500 – 480)	= 1800
Interest on bonds:	10 · 10000 · 4.5%	= 4500
Short rebate:	90 · 500 · 1.5%	= 675
Total:		= 4975 USD

In % the total return on capital is: 4975 USD / 90000 USD = **5,5278%**.

Scenario 4

Depreciation of bond price:	10 · (8800 – 9000)	= –2000
Depreciation of stock price:	90 · (500 – 480)	= 1800
Interest on bonds:	10 · 10000 · 4.5%	= 4500
Short rebate:	90 · 500 · 1.5%	= 675
Interest on borrowing	2% · 45000	= –900
Total:		= 4075 USD

In % the total return on capital is 4075 USD / 45000 USD = **9,0556%**.

Source: (Anson, 2006, p. 57-58).

The example shows how the strategy of volatility trading of convertible bonds works. It also explains why hedge fund managers leverage their positions–it simply brings them more absolute return for which they charge their clients the performance fee. It is their main motivation, so no wonder they are innovative and hire financial engineers to make hedge funds even more powerful and efficient. They make money even in situations of the price depreciation. However, we must be aware of the fact that in more dramatic scenarios (of much bigger prices' depreciation) the final outcome may be much more volatile.

7.3.3. Stock swap mergers with a collar in merger arbitrage

A *stock swap merger with a collar*, known also as *fixed rate stock swap merger*, is a complex merger situation where the exchange ratio is based on the price of the acquiring company at the date of completion or–in extreme cases–when the target company has the right to cancel negotiations about the merger, if the value of the share of the bidding company goes below a given value, called a *collar*.

The outcome of the merger depends on the price of shares of the acquiring company, therefore such a deal is highly sensitive to the volatility of the underlying stock. A hedge fund manager treats the collar as an option. The outcome of more complicated mergers is marked by a greater uncertainty and generally spreads are wider (Stefanini, 2006, p. 89). The following example shows the case.

Example 5

Stock swap merger with a collar

Company A wants to acquire a company B as follows:

- if share A is traded under 51 USD, company A offers 0.478 shares in exchange for 1 share B,
- if share A is traded between 51 USD and 63 USD, company A offers 22 USD for 1 share B (22 USD is a collar),
- if share A is traded above 63 USD, company A offers 0.379 shares in exchange for 1 share B.

Company A makes a collar bid to set the minimum and maximum number of own shares to be issued. The collar is defined based on the share price of the bidding company at the date of completion of the bid. Before the bid price is determined, a hedge fund manager has the opportunity to delta hedge by trading the option value.

Source: (Stefanini, 2006, p. 89).

The above characteristics and examples explain what the hedge funds are and how their managers can earn absolute returns. The complexity of their activity is a reasons why hedge funds are considered one of the most efficient financial innovations in the world. This is also a reason why they are offered only to high net worth investors who have enough experience or professional advice to be able to take part in this business. As the last decade showed this business can be very lucrative, but it can be also very volatile. Therefore the full understanding of the characteristics of hedge funds is so important. I hope this chapter brings the reader closer to it and helps her to find out how to profit (and not to lose) thanks to them. Only then hedge funds will be an efficient way of capital allocation and attractive form of the alternative investments.

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FINANCIAL ENGINEERING:

Methods and cases

This textbook contains materials for several courses which are taught in the Master's Programme in Financial Engineering that is run at the Poznań University of Economics and Business. The book consists of seven chapters that cover the main areas of quantitative finance: investment, financial instruments pricing, financial risk measuring and management as well as corporate finance.

The main part of the book is devoted to the mathematical models used in the field of finance. There are four chapters devoted to the pricing of financial instruments: from pricing equities using Capital Asset Pricing Model, through derivative instruments on equites, to more complicated derivatives on interest rates. The last topic is illustrated with some genuine examples from the markets in the post-crisis period. One chapter describes basic models and concepts used in measuring financial risk. Two other chapters are about investment. One describes the way in which companies finance their activities. The second one describes investment strategies of hedge funds. All chapters contain exercises and examples from the real markets.

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